Copula Models and Integrated Risk Management (Chapter -9)

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Learning Objectives

- Identify the shortcomings of normal and multivariate normal distributions in risk modeling.
- Explain why the normal assumption underestimates tail risk and joint extreme events.
- Evaluate how these limitations can distort portfolio diversification analysis.
- Recognize the need for alternative models (e.g., t-distributions, copulas) that better capture real-world dependence and risk.

Objectives

1. **Threshold correlations:** Define and plot to detect multivariate non-normality.
2. **Multivariate distributions:** Review standard normal; introduce symmetric and asymmetric t distributions.
3. **Copula modeling:** Define and develop the concept for linking marginals.
4. **Integrated risk management:** Apply copulas to assess and manage portfolio-wide risk.

Why it is Important?

- The normal distribution is mathematically convenient but poorly fits real asset returns.
- It underestimates large negative returns (fat tails and extreme events).
- The multivariate normal model similarly fails to capture joint extremes across assets.
- This leads to underestimation of portfolio risk during crises.
- Accurate risk modeling must account for tail dependence and non-normal behavior.
- **Alternative models** (e.g., t-distributions, copulas) that better capture real-world dependence and risk.

Threshold correlation

- Threshold correlation measures the correlation between two (or more) variables conditional on extreme events—for example, when both variables are below or above certain quantiles.
  - Detects **dependence** in the tails that a standard correlation might miss.
  - Reveals multivariate **non-normality**, because the multivariate normal distribution has constant correlation across all thresholds.
  - Higher threshold correlation in the tails indicates stronger joint extreme risk than the normal model predicts.
- **Use in Risk Management:**
  - Helps risk managers detect underestimation of joint extreme losses.
  - Guides the choice of alternative multivariate models (e.g., t-distributions, copulas) to capture realistic tail dependence.

Plotting

- Choose a set of thresholds q∈(0,0.5] for the lower tail (or q∈[0.5,1) for the upper tail).
- Compute conditional correlations ρ(q) at each threshold.
- Plot ρ(q) vs q

Interpretation:

- Flat line: consistent with multivariate normality.
- Increasing or decreasing curve: indicates non-normal tail dependence, signaling higher joint risk during extreme events.

Limitation of standard distributions:

- Multivariate normal distribution **underestimates threshold correlations** in daily returns.
- Multivariate t distribution allows **larger threshold correlations**, but having the **same degrees-of-freedom (d) for all assets is restrictive**.
- Asymmetric t distribution is **more flexible**, but requires **estimating many parameters simultaneously**.

The Copula Modeling Approach

- Developed in statistics to link different marginal distributions into a valid multivariate distribution.

- Allows flexible modeling of dependence separate from marginals.

$$F(x1,\ldots,xn)=C(F1(x1),\ldots,Fn(xn))$$

- Copulas **connect the marginal distributions** across assets to form a **multivariate density**.

Sklar’s Theorem:

- Any multivariate CDF with marginal can be expressed using a unique copula function G(.)

$$F(z_1, \ldots, z_n) = G(F_1(z_1), \ldots, F_n(z_n)) \\ = G(u_1, \ldots, u_n)$$

The $G(u_1, \ldots, u_n)$ function is sometimes known as the copula CDF.

Sklar's Theorem:

- The joint PDF can be written as:

$$f(z_1, ..., z_n) = g(u_1, ..., u_n) \prod_{i=1}^{n} f_i(z_i)$$

- The Copula PDF:

$$g(u_1, \ldots, u_n) \equiv \frac{\partial^n G(u_1, \ldots, u_n)}{\partial u_1 \cdots \partial u_n}$$

- The logarithm form:

$$\ln L = \sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t}) + \sum_{t=1}^{T} \sum_{i=1}^{n} \ln f_i(z_{i,t})$$

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Advantages:

- Simplifies high-dimensional modeling.
- Allows each asset to follow different univariate distributions (e.g., asymmetric t).
- Analogy with DCC/GARCH models: copulas allow combining different marginals like DCC allows **combining different GARCH models.**

Limitations:

- Sklar's theorem is very general: holds for a large class of distributions.
- It does not specify the functional form of G or g; modeling choices are required to implement copulas

Types of Copulas

Type	Description	Tail Dependence
Gaussian Copula	Correlation-based, symmetric	None
t-Copula	Uses t-distribution	Strong tail dependence
Clayton Copula	Captures lower-tail dependence	Left-tail
Gumbel Copula	Captures upper-tail dependence	Right-tail
Frank Copula	Symmetric, moderate dependence	None

Integrated Risk Management

- Assess and manage aggregate financial risk — across credit, market, liquidity, and operational domains.
- Key challenge: Risks are interdependent, not additive.
- Requires multivariate modeling: capturing both marginal distributions (individual risks) and dependence structure (how they move together).

Marginal Distributions of Risk Factors

- Financial variables rarely follow normal distributions.
- **Heavy tails** → extreme events more likely than under Normal.
- **Skewness** → asymmetric losses/gains.
- **Common choices:**

Risk Type	Typical Distribution	Features
Market returns	Student-t, skew-t	Fat tails, skewness
Credit losses	Lognormal, Gamma	Non-negative, right-skewed
Operational losses	Pareto, Weibull	Heavy tails
Insurance claims	Lognormal	Skewed & fat-tailed

Applications in Integrated Risk Management

- Portfolio Risk Aggregation
  - Combine credit, market, and operational risk distributions.
  - Simulate joint losses via copulas.
- Economic Capital Estimation
  - Compute Value-at-Risk (VaR) and Expected Shortfall (ES) at firm level.
- Stress Testing
  - Explore impact of extreme co-movements.
- Credit Portfolio Models
  - Gaussian/t-copulas in models like CreditMetrics, CreditRisk+.

Example: t-Copula for Portfolio Losses

- Step 1: Fit Student-t marginals to individual loss series.
- Step 2: Estimate correlation matrix and degrees of freedom.
- Step 3: Simulate dependent uniforms via t-copula, then invert marginals.
- Step 4: Aggregate to portfolio loss distribution.
- Captures joint tail risk better than Gaussian copula.

Advantages of Copula Approach

- Separates marginals and dependence modeling.
- Flexible across risk types.
- Captures tail dependence — vital for systemic risk modeling
- Enhances accuracy in VaR and ES aggregation.

Limitations

- Parameter estimation can be complex.
- Choice of copula impacts tail behavior strongly.
- Dynamic (time-varying) dependence not easily handled by static copulas.
- High-dimensional copulas (many risks) computationally demanding.

Risk Management Using Copula Models

❑**Copula VaR and ES by Simulation:**

➢**Monte Carlo simulation** is used to compute portfolio VaR and ES from copulas.

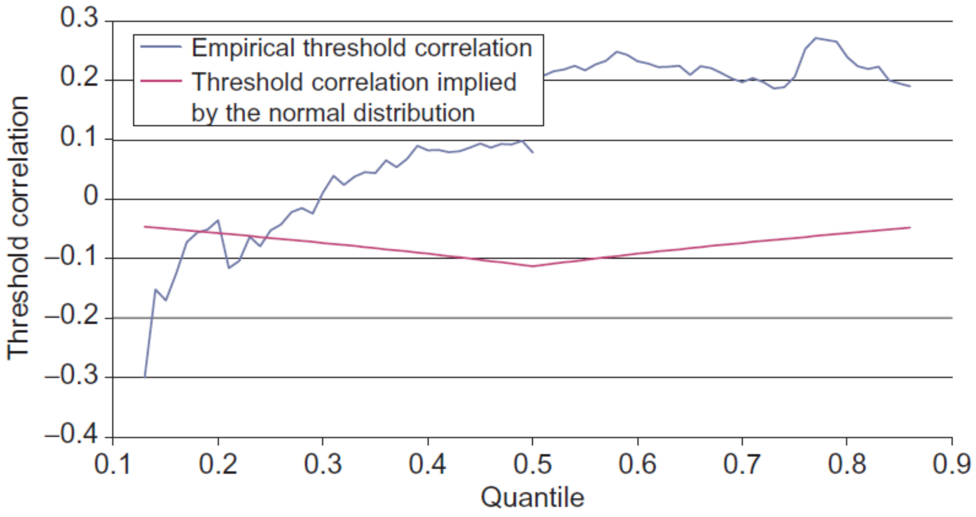
❑**Copula model building steps:**

- Estimate **dynamic volatility** for each asset: $z_{i,t} = r_{i,t}/\sigma_{i,t}$.
- Estimate **marginal density** for each asset: $u_{i,t} = F_i(z_{i,t})$
- Estimate **copula parameters** via log-likelihood: $\sum_{t=1}^{T} \ln g(u_{1,t}, \ldots, u_{n,t})$.

❑**Monte Carlo simulation (reverse steps):**

- Simulate **copula probabilities**: $(u_{1,t}, \ldots, u_{n,t})$
- Convert to **shocks** using marginal inverse CDF: $z_{i,t} = F_i^{-1}(u_{i,t})$
- Convert shocks to **returns**: $r_{i,t} = \sigma_{i,t} z_{i,t}$

Figure 9.1 Threshold correlation for S&P 500 versus 10-year treasury bond returns.



- ➢Patterns in Figure 9.2 differ notably from Figure 9.1.
- ➢Higher threshold correlations occur when both shocks are negative than when both are positive.
- ➢Indicates significant nonlinear left-tail dependencies between stocks and bonds.
- ➢Bivariate normal distribution poorly captures the empirical threshold correlations.

Figure 9.2 Threshold correlation for S&P 500 versus 10-year treasury bond GARCH shocks.

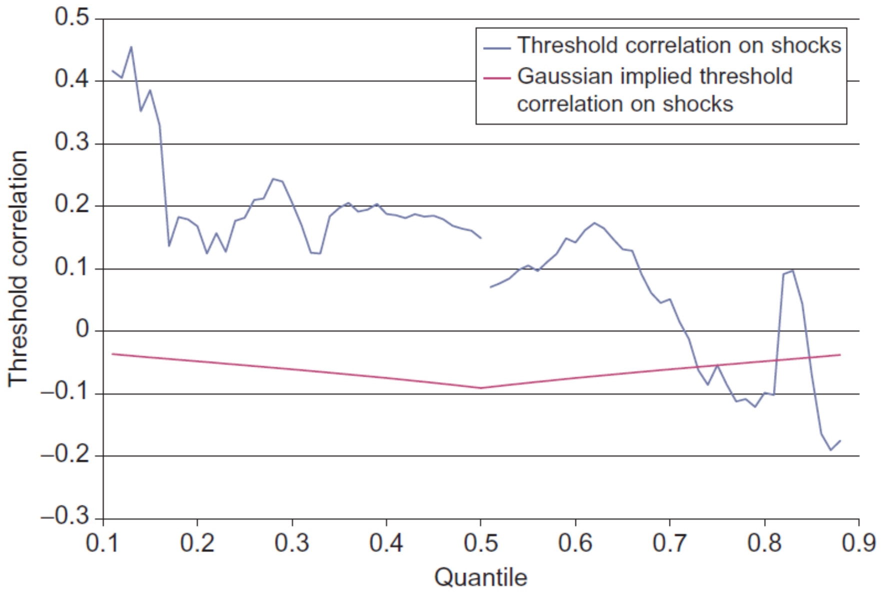


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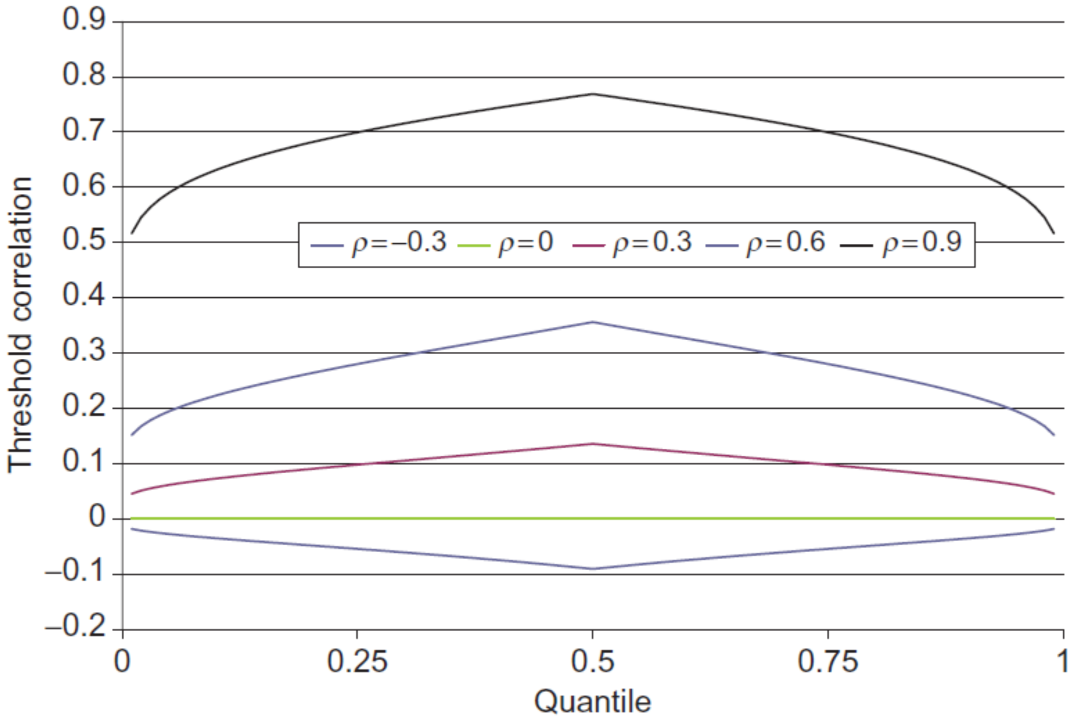
Figure 9.3 – Bivariate Normal Distribution:

- Threshold correlations vary symmetrically across quantiles.
- Peak correlations occur near the median quantile (0.5), decreasing toward the tails.
- Lower correlations appear in the extreme tails for all values of linear correlation (ρ).
- The pattern reflects the limitations of linear correlation models, as extreme events (tails) are underrepresented in their dependency.

Figure 9.3 Simulated threshold correlations from bivariate normal distributions with various linear correlations.



Notes: The threshold correlations from the bivariate normal distribution are plotted for various values of the linear correlation parameter.

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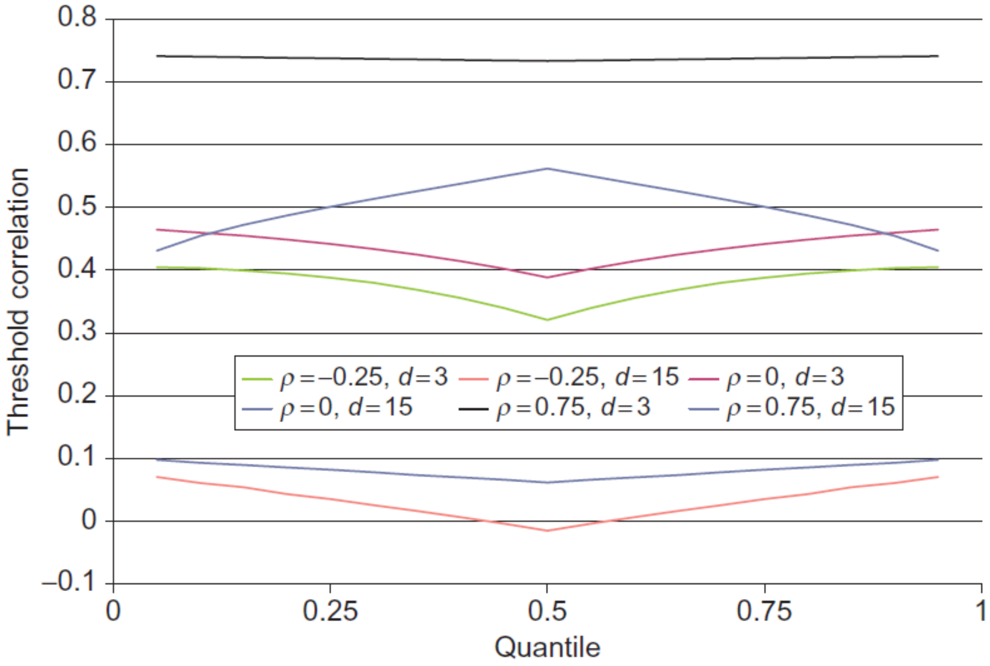
Figure 9.4 – Bivariate Symmetric t Distribution:

- Threshold correlations are higher in the tails compared with the normal distribution, especially for smaller degrees of freedom (d).
- Correlations are flatter across quantiles, showing stronger tail dependence.
- Lower d values amplify tail dependence, while higher d values reduce it, approaching the normal-like pattern.
- Captures nonlinear dependence, particularly in extreme negative and positive outcomes.

Figure 9.4 Simulated threshold correlations from the symmetric *t* distribution with various parameters.



Notes: We simulate a large number of realizations from the bivariate symmetric *t* distribution. The figure shows the threshold correlations from the simulated data when using various values of the correlation and *d* parameters.

Summary

- Multivariate risk models rely on assumptions about the joint (multivariate) distribution of return shocks.
- The multivariate normal distribution is computationally convenient but underestimates extreme dependence between assets.
- Threshold correlation is a useful measure of extreme dependence observed in asset returns and in candidate multivariate models.
- The multivariate symmetric t and especially the asymmetric t distribution provide stronger tail dependence (higher threshold correlations) than the normal model.
- However, the asymmetric t distribution becomes complex and difficult to estimate in high-dimensional settings.
- Copula models allow for flexible combination of different marginal distributions with customized dependence structures.
- The normal and t copulas are practical, flexible, and perform well even in high-dimensional applications.
- Copulas are particularly useful for integrated risk management, enabling consistent linking of risk models from multiple business units into a coherent, organization-wide risk measure.

Example:

We observe 2 assets (A and B) with 3 observations of returns:

Observation	Asset A Return	Asset B Return
1	0.02	0.03
2	0.05	0.01
3	-0.01	0.04

1. Compute Marginal Probabilities
2. Gaussian Copula Transformation
3. Calculate Copula density for observation 1
4. Interpret your results.
5. Visualise
6. Change ρ to 0.9. Recompute the copula. How does dependence change?

Note: c>1 means positive dependence between assets (more likely joint extreme values).

c<1 would indicate weaker or negative dependence.

Thank You