Correlation Modeling

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Learning Objectives

- Explain the concept of correlation between asset returns.
- Understand how correlation affects portfolio risk.
- Compute Value at Risk (VaR) for simple portfolios.
- Explain the concept of conditional covariance and correlation.
- Understand quasi-maximum likelihood estimation (QMLE).
- Discuss realized and range-based covariance estimators.

Portfolio Risk and Correlation

Portfolio risk depends not only on the risk of individual assets but also on how they move together — **their correlation**.

For a portfolio of two assets:

$$R_P = w_1 R_1 + w_2 R_2$$

The **portfolio variance** is:

$$\sigma_p^2 = w_1^2 \sigma_1^2 + w_2^2 \sigma_2^2 + 2 w_1 w_2 \sigma_{12}$$

Where,

$\sigma_{12} = \rho_{12} \sigma_1 \sigma_2$

$\rho_{12} =$ correlation coefficient between R_1 and R_2

Portfolio Variance and Covariance

For daily log returns the portfolio return relationship will hold approximately

$$R_{PF,t+1} \approx \sum_{i=1}^{n} w_{i,t} R_{i,t+1}$$

where the sum is taken over the *n* securities in the portfolio, $w_{i,t}$ denotes the relative weight of security *i* at the end of day *t*.

The **variance** of the portfolio can be written as

$$\sigma^2_{PF,t+1} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{i,t} w_{j,t} \sigma_{ij,t+1} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_{i,t} w_{j,t} \sigma_{i,t+1} \sigma_{j,t+1} \rho_{ij,t+1}$$

Value at Risk (VaR) for Simple Portfolios

- **VaR is the maximum loss not exceeded with a given probability α over a specific time horizon.**

For a simple portfolio with normally distributed returns:

$$VaR_\alpha = z_\alpha \cdot \sigma_p \cdot V_0$$

Where,

- z_α: quantile of standard normal distribution
- σ_p: portfolio standard deviation (measure of risk or volatility).
- V_0: current portfolio value

Example:

If $V_0 = 1{,}000{,}000$; $\sigma_p = 0.02$; and $\alpha = 0.05$:

$$VaR_{0.05} = 1.65 \times 0.02 \times 1000000 = 32{,}900$$

There's a 5% chance the portfolio will lose more than $32,900 in one day.

Value at Risk (VaR) for Simple Portfolios

If we are willing to assume that returns are multivariate normal, then the portfolio return, which is just a linear combination of asset returns, will be normally distributed, and we have

$$\boldsymbol{VaR_{t+1}^{p} = -\sigma_{PF,t+1}\,\varphi_{p}^{-1}}$$

It gives the **potential maximum loss** at a certain confidence level "p" for the portfolio at time t+1.

Where,

- VaR_{t+1}^{p}: Value at Risk for the portfolio PF at time t+1 and confidence level p
- $\sigma_{PF,t+1}$: **Forecasted portfolio standard deviation (volatility)** at time t+1. It measures how much the portfolio's return is expected to fluctuate.
- φ_{p}^{-1}: The inverse cumulative distribution function (quantile) of the standard normal distribution corresponding to the probability p. It gives the critical value (z-score) for the desired confidence level.
  - For 95% confidence: $\varphi_{0.05}^{-1} = -1.645$
  - For 99% confidence: $\varphi_{0.01}^{-1} = -2.33$
  - **Multiplying this by the portfolio's volatility $\boldsymbol{\sigma_{PF,t+1}}$ gives the expected loss, and the negative sign makes it a positive risk measure.**
- Negative sign (-): Ensures VaR is expressed as a positive loss value, since the quantile for the left tail (loss) is negative.

Value at Risk (VaR) for Simple Portfolios

- Example:

$$VaR_{t+1}^{p} = -(0.02)(-1.645) = 0.0329$$

If the portfolio value is $V_0 = 1{,}000{,}000$:

$$VaR = 0.0329 \times 1000000 = 32{,}900$$

There's a 5% chance the portfolio will lose more than $32,900 in one day.

Expected Shortfall (ES)

➢ measures the **average loss in the worst p% of cases**:

$$ES_{t+1}^{p} = \sigma_{PF,t+1} \frac{\emptyset(\varphi_p^{-1})}{p}$$

Where,

- ES_{t+1}^{p}: Expected Shortfall at confidence level 1-p
- $\sigma_{PF,t+1}$: Portfolio volatility
- φ_p^{-1}: Inverse standard normal CDF
- $\emptyset(\varphi_p^{-1})$: PDF value at that quantile
- p: Tail probability

Expected Shortfall (ES)

- Example: For the previous example, we can calculate ES as:

$$ES_{t+1}^{0.05} = 0.02 \times \frac{\emptyset(-1.645)}{0.05} = 0.02\frac{0.103}{0.05} = 0.0412 = 4.12\%$$

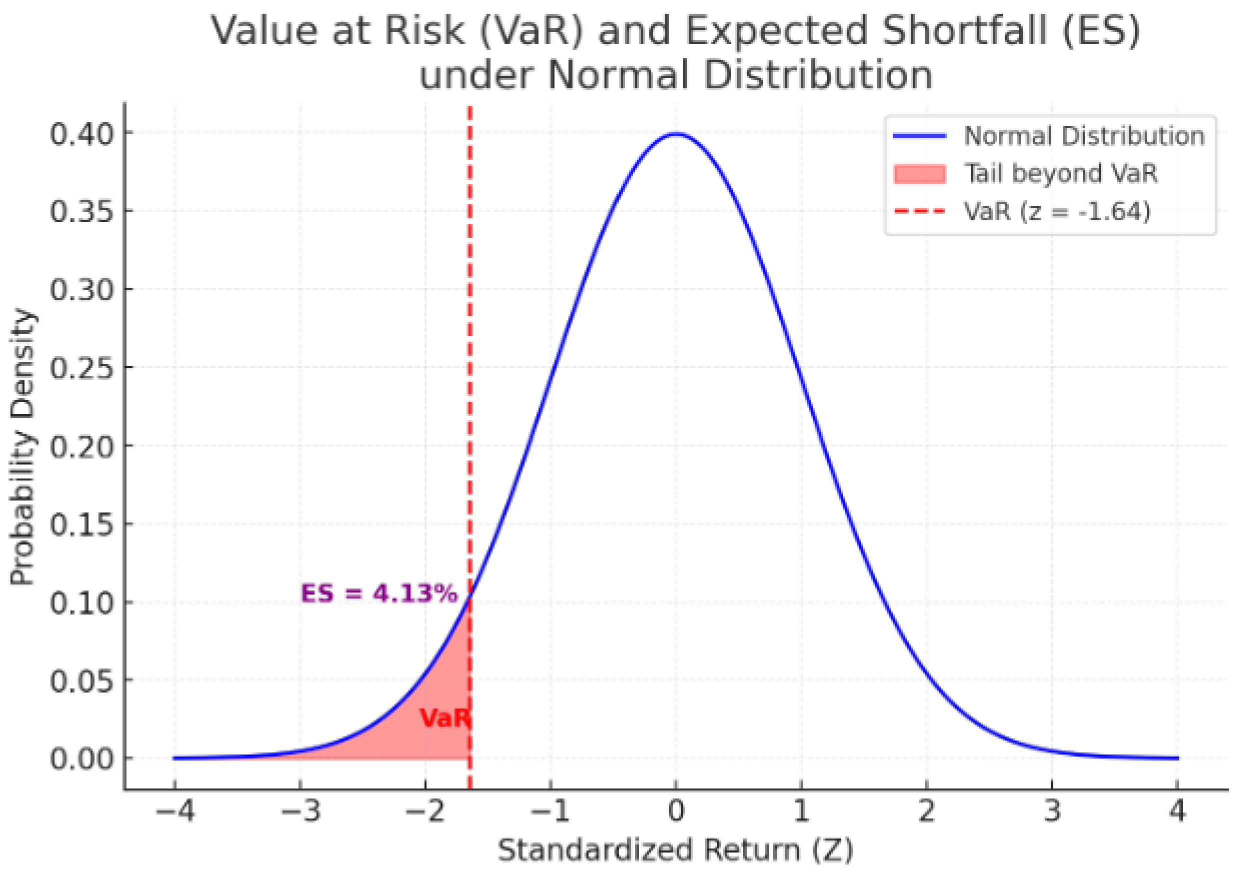
If your portfolio value is \$1,000,000:

$$ES_{t+1}^{0.05} = 0.0412 \times 1000000 = \$41{,}200$$

If you breach that level, the average loss will be about \$41,200.

VaR vs ES difference

- VaR gives the threshold loss not exceeded 95% of the time.
- ES gives the average loss beyond that threshold.
- The blue curve shows the standard normal distribution of returns.
- The red dashed line marks the VaR (Value at Risk) — the cutoff point where the worst 5% of outcomes begin.
- The red shaded area represents those extreme losses (the 5% tail).
- The purple label (ES = 4.13%) shows the Expected Shortfall, which is the average loss within that red tail region.



Limitations

- **Volatility forecasts alone are not enough**
  - Even if we can accurately predict individual asset volatilities (risk of each stock), we still need to know how those assets move together — i.e., their correlations.
  - Portfolio risk depends not only on each asset's volatility but also on how they co-move.
- **The number of correlations grows rapidly**
  - For n assets, the number of unique pairwise correlations is:

  Example:
  If n=100 then we'll have 4950 correlations to model, which would be a daunting task.
- **The problem is "high-dimensional"**
  - As portfolios grow large, modeling thousands of correlations becomes computationally complex and unstable.
  - Traditional methods (like simple pairwise correlations) are not practical.
- **Need for large-dimensional modeling techniques**
  - Factor models
  - Shrinkage estimators
  - Dynamic Conditional

Exposure Mapping

- In large portfolios, directly modeling all pairwise correlations is computationally intensive.
- To simplify this, we can reduce dimensionality by expressing the portfolio return as a function of a few key risk factors.
- This is known as Exposure Mapping or Factor Structure Modeling.
- Exposure mapping is a **dimension-reduction technique** in portfolio risk modeling.
- It links portfolio risk to **common risk factors**.

Exposure Mapping

➢ **Basic One Factor Model:**

$$r_{PF,t+1} = r_{MKt,t+1} + \varepsilon_{t+1}$$

where:

- $r_{PF,t+1}$: portfolio return
- $r_{MKt,t+1}$: market (systematic) return
- ε_{t+1}: idiosyncratic (asset-specific) risk, assumed independent of the market

Exposure Mapping

➢ Portfolio Variance Decomposition

- The total portfolio variance is:

$$\sigma^2_{PF,t+1} = \sigma^2_{MKt,t+1} + \sigma^2_{\epsilon}$$

Where,

- The first term represents systematic market risk.
- The second term represents idiosyncratic risk (unique to the portfolio).

Exposure Mapping

- **Index Mapping:**

- In a highly diversified portfolio (e.g., similar to the **S&P 500 index return**), no correlation modeling is necessary. we can assume:

$$\sigma^2_{PF,t+1} \approx \sigma^2_{MKt,t+1}$$

- **VaR under index mapping:**

- Assuming normality,

$$VaR^p_{t+1} = -\sigma_{MKt,\,t+1}\,\varphi_p^{-1}$$

Exposure Mapping

- **Beta Mapping:**

- In many cases, the portfolio's return may not move exactly one-to-one with the market. We introduce a **sensitivity parameter (β)** to capture this relationship:

$$r_{PF,t+1} = \beta r_{MKt,t+1} + \varepsilon_{t+1}$$

Hence,

$$\sigma^2_{PF,t+1} = \beta^2 \sigma^2_{MKt,t+1} + \sigma^2_{\epsilon}$$

If the portfolio is well diversified and the market explains most of the variation:

$$\sigma^2_{PF,t+1} \approx= \beta^2 \sigma^2_{MKt,t+1}$$

Exposure Mapping

- **Multi-Factor Exposure Model:**

For large, diversified portfolios, risk may depend on multiple systematic factors — such as:

- Country equity indices
- Foreign exchange rates
- Commodity price indices

- We can express portfolio return as:

$$r_{PF,t+1} = \beta_1 r_{F1,\,t+1} + \beta_2 r_{F2,\,t+1} + \ldots + \beta_{n_F} r_{n_F,\,t+1} + \varepsilon_{t+1}$$

where n_F is the number of factors, typically much smaller than the number of assets (n_F<<n) and ε_{t+1} are assumed to be independent of the risk factors.

Exposure Mapping

➢ **Portfolio Variance under Factor Model**

➢ The portfolio variance can be written as:

$$\sigma^2_{PF,t+1} = \beta_F' \Sigma_{F,t+1} \beta_F + \sigma^2_\epsilon$$

Where,

- $\Sigma_{F,t+1}$: covariance matrix of factor returns
- β_F: vector of factor exposures

➢ If the factor model captures most of the systematic variation:

$$\sigma^2_{PF,t+1} \approx \beta_F' \Sigma_{F,t+1} \beta_F$$

This drastically **reduces dimensionality**, since we now model variances and correlations among **a few factors**, not thousands of individual assets.

GARCH Conditional Covariance

- Portfolios often include many assets ($n \geq 10$) or risk factors.
- Risk managers must estimate an $n \times n$ covariance matrix to capture how asset returns move together.
- Goal: Model time-varying covariances that evolve with market conditions

GARCH Conditional Covariance

❑Rolling Covariance Estimation:

- **Simple approach:** Use rolling averages over a fixed window of *m* days.
- For the covariance between asset (or risk factor) *i* and *j*; we can simply estimate,

$$\sigma_{ij,t+1} = \frac{1}{m}\sum_{\tau=1}^{m} R_{i,t+1-\tau} R_{j,t+1-\tau}$$

- **Pros:** Easy to compute.
- **Cons:**
  - Sensitive to choice of window length m.
  - Equal weighting of old and new data → not adaptive.

GARCH Conditional Covariance

❑ Exponential Smoothing (RiskMetrics Approach):

- Gives **more weight to recent data**:

$$\sigma_{ij,t+1} = (1-\lambda) R_{i,t} R_{j,t} + \lambda \sigma_{ij,t}$$

where typically λ=0.94.

- **Advantages:** Smooth, adaptive estimate.
- **Limitations:**
  - No mean reversion — if covariance falls, it stays low indefinitely.
  - Same decay parameter λ must be used across all asset pairs for matrix consistency.

GARCH Conditional Covariance

GARCH-style Covariance Modeling:

- To allow **mean reversion**, use a **GARCH(1,1)-type** model for covariance:

$$\sigma_{ij,t+1} = \omega_{ij} + \alpha R_{i,t} R_{j,t} + \beta \sigma_{ij,t}$$

- Long-run mean covariance:

$$\sigma_{ij} = \omega_{ij} / (1 - \alpha - \beta)$$

Where,

α: sensitivity to new shocks

β: persistence of past covariance

w_{ij}:baseline level ensuring mean reversion

GARCH Conditional Covariance

Internal Consistency (Positive Semi-Definiteness):

- A valid covariance matrix must satisfy:

$$w_t' \Sigma_{t+1} w_t \geq 0$$

- This corresponds to saying that the covariance matrix is positive semidefinite.
- It is ensured by estimating volatilities and covariance's in an internally consistent fashion.

Limitations:

- Unfortunately, it is not clear that the persistence parameters α, β and λ should be the same for all variances and covariance.
- Leads to more flexible **multivariate GARCH frameworks** like:
  - **BEKK model**
  - **DCC-GARCH model**

Example:

Day	Asset A	Asset B
1	0.20	0.10
2	-0.30	-0.25
3	0.15	0.05
4	0.40	0.35
5	-0.10	-0.20
6	0.25	0.15
7	-0.35	-0.30
8	0.10	0.05
9	-0.05	0.00
10	0.30	0.20

1. Calculate mean and standard deviation, of each asset.
2. Calculate correlation between Asset A and Asset B
3. Calculate the unconditional 1-day, 1% Value-at-Risk (VaR) for each asset individually, assuming normal distribution:

$$VaR_{i,1\%} = -(\mu_i + z_{0.01}\sigma_i).$$

4. Portfolio VaR (50%-50%):

- Portfolio mean:

$$\mu_p = 0.5\mu_A + 0.5\mu_B$$

- Portfolio Variance:

$$\sigma_P^2 = 0.5^2\sigma_A^2 + 0.5^2\sigma_B^2 + 2(0.5)(0.5)\sigma_A\sigma_B$$

- Portfolio VaR:

$$VaR_{p,1\%} = -(\mu_p + z_{0.01}\sigma_p).$$

Next…

- **Dynamic Conditional Correlation (DCC)**
- **Exponential Smoother Correlations**
- **Mean-Reverting Correlation**
- **QMLE**
- **Realized and Range-Based Covariance**

Dynamic Conditional Correlation (DCC)

- Capture time-varying correlations between multiple asset returns.

- Key Features:

- Models conditional variances using univariate GARCH for each asset.
- Models conditional correlations dynamically over time.
- Provides a flexible and realistic correlation structure, unlike static correlations.

Dynamic Conditional Correlation (DCC)

correlation is defined from covariance and volatility by

$$\rho_{ij,t+1} = \sigma_{ij,t+1} / (\sigma_{i,t+1} \sigma_{j,t+1})$$

we have the RiskMetrics model, then

$$\sigma_{ij,t+1} = (1-\lambda) R_{i,t} R_{j,t} + \lambda \sigma_{ij,t}, \quad \text{for all } i, j$$

then we get the implied dynamic correlations

$$\rho_{ij,t+1} = \frac{(1-\lambda) R_{i,t} R_{j,t} + \lambda \sigma_{ij,t}}{\sqrt{((1-\lambda) R_{i,t}^2 + \lambda \sigma_{i,t}^2)((1-\lambda) R_{j,t}^2 + \lambda \sigma_{j,t}^2)}}$$

Dynamic Conditional Correlation (DCC)

- We can then standardize each return by its dynamic standard deviation to get the standardized returns,

$$z_{i,t+1} = R_{i,t+1}/\sigma_{i,t+1} \text{ for all } i$$

- Modeling the conditional correlation of the raw returns is equivalent to modeling the conditional covariance of the standardized returns.

$$\begin{aligned}
E_t\left(z_{i,t+1}z_{j,t+1}\right) &= E_t\left(\left(R_{i,t+1}/\sigma_{i,t+1}\right)\left(R_{j,t+1}/\sigma_{j,t+1}\right)\right) \\
&= E_t\left(R_{i,t+1}R_{j,t+1}\right)/(\sigma_{i,t+1}\sigma_{j,t+1}) \\
&= \sigma_{ij,t+1}/(\sigma_{i,t+1}\sigma_{j,t+1}) \\
&= \rho_{ij,t+1}, \text{ for all } i,j
\end{aligned}$$

Exponential Smoother Correlations

Exponential smoothing is a method to give **more weight to recent observations** while gradually reducing the influence of older data.

Define the smoothed covariance-like quantity, which evolves over time as:

$$q_{ij,t+1} = (1-\lambda)\left(z_{i,t}z_{j,t}\right) + \lambda q_{ij,t}, \quad \text{for all } i, j$$

Where:

λ = smoothing parameter (0<λ<1)

$z_i z_j$= cross-product of standardized returns (instantaneous correlation contribution)

q_{ij}= previous smoothed value

Recent observations have weight $1-\lambda$

Past smoothed correlations decay with weight λ

Exponential Smoother Correlations

❑Normalization to Obtain Conditional Correlation

The $q_{ij,t+1}$ matrix is not guaranteed to have values between -1 and 1. To obtain the **actual conditional correlation** we normalize:

$$\rho_{ij,t+1} = \frac{q_{ij,t+1}}{\sqrt{q_{ii,t+1} q_{jj,t+1}}}$$

Where:

$q_{ii,t+1}$ = smoothed variance of standardized returns of asset i

$q_{jj,t+1}$ = smoothed variance of standardized returns of asset j

➢**Why normalization?**

- Correlations must satisfy $-1 < \rho_{jj,t+1} < 1$
- Normalizing ensures the smoothed matrix behaves like a proper **correlation matrix**.

Exponential Smoother Correlations

- **Choosing the Smoothing Parameter λ**
  - Common choices: $\lambda = 0.94$ for daily financial returns (as suggested by RiskMetrics)
  - Lower $\lambda \rightarrow$ smoother correlation, slower to react to market changes
  - Higher $\lambda \rightarrow$ more responsive, but may be noisy

Mean-Reverting Correlation

Correlations between asset returns are often not only time-varying but also tend to revert toward a long-term average over time.

This phenomenon is called mean reversion.

$$\rho_{ij,t+1} = \frac{q_{ij,t+1}}{\sqrt{q_{ii,t+1} q_{jj,t+1}}}$$

$$q_{ij,t+1} = \bar{\rho}_{ij} + \alpha\left(z_{i,t} z_{j,t} - \bar{\rho}_{ij}\right) + \beta\left(q_{ij,t} - \bar{\rho}_{ij}\right)$$

$$\bar{\rho}_{ij} = \frac{1}{T} \sum_{t=1}^{T} z_{i,t} z_{j,t}$$

Realized Covariance

- Traditional covariance uses daily or lower-frequency returns, which may miss intraday variation.
- Realized covariance uses high-frequency intraday returns to provide a more accurate estimate of the actual covariance over a day.
- Why it matters:
  - Provides better risk estimates
  - Captures intraday dynamics and volatility clustering
  - Useful as an input for realized-GARCH or DCC models

$$RCov_{12,t+1}^{m} = \sum_{j=1}^{m} R_{1,t+j/m} R_{2,t+j/m}$$

$$\rho_{12,t+1}^{m} = RCov_{12,t+1}^{m} / \sqrt{RV_{1,t+1}^{m} RV_{2,t+1}^{m}}$$

Go through…

- **Bivariate Quasi Maximum Likelihood Estimation**
- **Composite Likelihood Estimation in Large Systems**
- **An Asymmetric Correlation Model**
- **Range-Based Covariance Using No-Arbitrage Conditions**

Thank You