Option Pricing (Chapter -10)

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Learning Objectives

- **What is Option Pricing?**
- **Call Option**
- **Out Option**
- **Payoff**
- **The Binomial Tree Model**
- **Example**

Option Pricing

- **Option pricing** means **finding the fair value (price)** of an **option contract** — either a **call** or a **put**.
- **Options** are financial derivatives that give the holder the *right*, but not the *obligation*, to buy or sell an underlying asset at a predetermined price within a specific period.
- Option pricing = figuring out **how much the right to buy or sell something in the future is worth today**.
- **Types of Options**
  - **Call Option:** Gives the right to **buy** an asset at a fixed price (the *strike price*).
  - **Put Option:** Gives the right to **sell** an asset at a fixed price.

European Call Option

Gives the right to **buy** one unit of the underlying asset $\tilde{T}$ **days** from now at price **X** (the *strike price*).

Call Payoff: $\mathbf{Max}\ \{S_{t+\tilde{T}} - \mathbf{X}, \mathbf{0}\}$

Symbol	Meaning
c	Current price of the call option
S_t	Current price of the underlying asset
$S_{t+\tilde{T}}$	Price of the asset at maturity
$\tilde{T}$	Days to maturity
X	Strike price

- The **"Price of the asset at maturity"** means the **market value of the underlying asset (such as a stock, commodity, or index) on the date when the option or contract expires**.
- The **strike price** (also called the **exercise price**) is the **fixed price** at which the holder of an option can **buy or sell** the underlying asset **when the option expires (at maturity)**.

European Put Option

Gives the right to **sell** one unit of the underlying asset $\tilde{T}$ **days** from now at the **strike price (X)**.

Put Payoff: Max $\{X - S_{t+\tilde{T}}, 0\}$

Symbol	Meaning
p	Current price of the put option
S_t	Current price of the underlying asset
$S_{t+\tilde{T}}$	Price of the asset at maturity
$\tilde{T}$	Days to maturity
X	Strike price

Payoff — The Core of Option Value

- The payoff function is the option's defining characteristic.
- The payoff function defines the value of the option at maturity.
- Depends on the underlying price at maturity,

Option Type	Payoff Function	Interpretation
Call	$Max\ \{S_{t+\tilde{T}}\ - X, 0\}$	Exercise if $S_{t+\tilde{T}} > X$
Put	$Max\ \{X\ -\ S_{t+\tilde{T}}, 0\}$	Exercise if $S_{t+\tilde{T}} < X$

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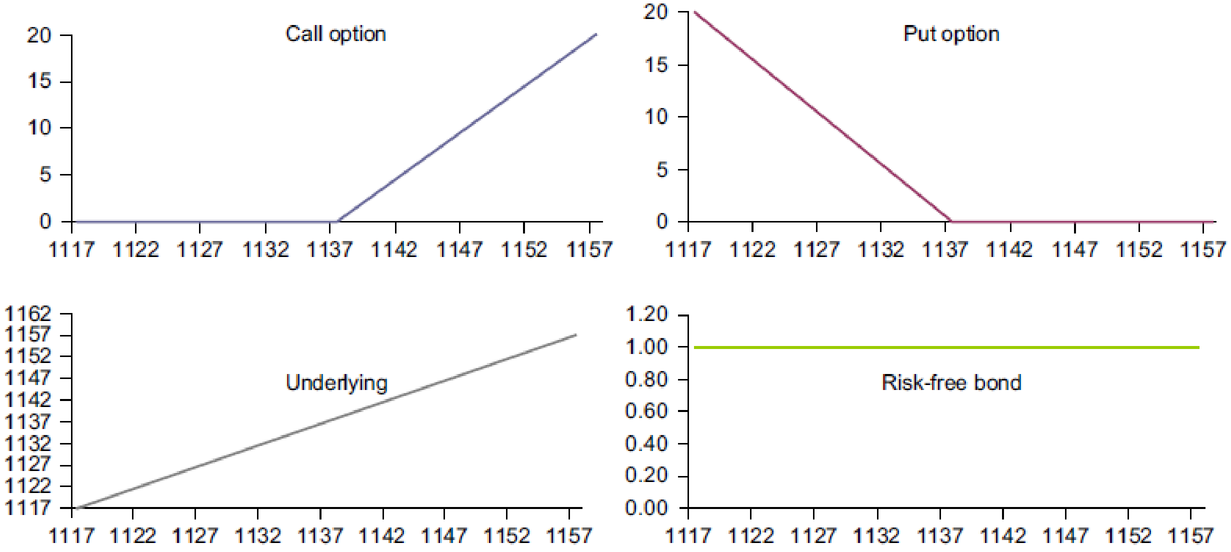
Figure 10.1 — Payoff as a Function of the Underlying Asset's Value
(Call Option, Put Option, Underlying Asset, and Risk-Free Bond)



Notes: All panels have the future value of the underlying asset on the horizontal axis. The top-left panel plots the call option value, the top right plots the put option value, the bottom left plots the underlying asset itself, and the bottom right plots the risk-free bond.

- The horizontal axis: future price of the underlying asset at maturity
- The vertical axis: payoff (value) of each instrument at maturity.
- The strike price (X = 1137) is the turning point where the option starts to gain value.
- Call = profit when price rises,
- Put = profit when price falls.
- Call options benefit from price increases.
- Put options benefit from price decreases.

Call Option (Top-Left Graph)

- **A call option gives the right to buy the underlying asset at a fixed price (strike price X = 1137).**
- Observation:
  - When the asset price < 1137, the line is flat at 0 → no profit
  - When the asset price > 1137, the line rises upward → profit increases as the price rises.

Formula:

$$\textbf{Payoff} = \mathbf{Max}\ \{S_{t+\tilde{T}} - \mathbf{X}, \mathbf{0}\}$$

Put Option (Top-Right Graph)

- **A put option gives the right to sell the underlying asset at price X = 1137.**
- Observation:
  - When the asset price > 1137, the line is flat at 0 → no gain
  - When the asset price < 1137, the line slopes downward → profit increases as price falls.

Formula:

$$\boldsymbol{Payoff} = \mathbf{Max}\,\{\mathbf{X} - \boldsymbol{S}_{\mathrm{t}+\widetilde{T}}, \mathbf{0}\}$$

Underlying Asset (Bottom-Left Graph)

- This shows the value of the asset itself.
- Observation:
  - The line rises steadily — value increases one-to-one with price.
- Formula:

$$\boldsymbol{Payoff = S_{t+\tilde{T}}}$$

Risk-Free Bond (Bottom-Right Graph)

- A risk-free bond gives a fixed return regardless of market movements (e.g., a Treasury bond).
- Observation:
- The line is perfectly flat — value stays constant over all asset prices.
- Formula:

$$\boldsymbol{Payoff = (1 + r)^T}$$

where r = risk-free rate, T = time to maturity.

Models

We will explore key models used in **European option pricing**, moving from classical to more advanced approaches that capture the complexities of real-world financial markets.

1. The Binomial Tree Model
2. The Black–Scholes–Merton (BSM) Model
3. The Gram–Charlier (GC) Expansion Model
4. GARCH-Based Option Pricing Models
5. Implied Volatility Function (IVF) Approach

The Binomial Tree Model

- The **key challenge** in option pricing is that the **future price of the underlying asset (stock)** is uncertain.
- To make the problem simpler, we assume a **binomial distribution** for future prices — meaning:
  - In a short time interval, the stock price can only move **up (u)** or **down (d)**.
- This is the simplest possible assumption about uncertainty.

Example

We want to find the **fair value** of a **call** and **put** option with:

Parameter	Symbol	Value
Current stock price	S_{t}	$1,000
Strike price	X	$900
Time to maturity	T	3 months (0.25 years)
Volatility (annualized)	σ	0.60 (60%)
Risk-free interest rate	r_f	5% per year
Steps in tree		2

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➢ Step 1: Build the Binomial Tree for Stock Price (Lecture)

Step	Description	Stock Price
A	Start	\$1,000
B	One Up	\$1,000 × 1.2363 = \$1,236.31
C	One Down	\$1,000 × 0.8089 = \$808.86
D	Up–Up	\$1,000 × $(1.2363)^2$ = \$1,528.47
E	Up–Down or Down–Up	\$1,000 × (1.2363 × 0.8089) = \$1,000
F	Down–Down	\$1,000 × $(0.8089)^2$ = \$654.25

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Using these up and down factors the tree is built from the current price of $1000 on the left side to three potential values in three months, namely $**1528.47** if the stock price moves up twice, $**1000** if it has one up and one down move, and $**654.25** if it moves down twice.

Table 10.1 Building the binomial tree forward from the current stock price

Market Variable				D
$S_t =$	1000			1528.47
Annual $r_f =$	0.05			
Contract Terms				
$X =$	900		B	
$T =$	0.25		1236.31	
Parameters				
Annual Vol =	0.6			
tree steps =	2	A		E
$dt =$	0.125	1000.00		1000.00
$u =$	1.23631111			
$d =$	0.808857893			
			C	
			808.86	
				F
				654.25

Notes: We construct a two-step binomial tree from today's price of $1,000 using an annual volatility of 60%. The total maturity of the tree is three months.

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Step 2: Compute Option Payoffs at Maturity

- Call Option Payoff: **Call Payoff: $\mathbf{Max}\ \{S_{t+\tilde{T}} - X, 0\}$**

Node	Stock Price	Payoff
D	$1,528.47	$628.47
E	$1,000.00	$100.00
F	$654.25	$0

- Put Option Payoff: **Put Payoff: $\mathbf{Max}\ \{X - S_{t+\tilde{T}}, 0\}$**

Node	Stock Price	Payoff
D	$1,528.47	$0
E	$1,000.00	$0
F	$654.25	$245.75

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- Step 3: Work Backward to Find Today's Option Price (**Lecture)**
- Step 4: Put Option Value (**Lecture)**

Fair value of Call Option = $181.47

- **Calculate *Risk Neutral Valuation (From Book)***

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In the **binomial option pricing model**, the **core idea** is to **create a risk-free portfolio** — a combination of stock and option that has **the same value no matter what happens** (stock goes up or down).

That's why we **choose Δ (the hedge ratio)** .

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- The Binomial tree approach is one of the most intuitive ways to introduce option pricing. It models the underlying asset price using a simple **up–down movement** at each step.
- However, the method has an important limitation:
  - It does not yield a **closed-form formula** for option prices.
  - We must compute prices numerically by working backward through the tree.
- To obtain a closed-form solution, we assume asset returns are i.i.d. normal.
- Assuming normal returns leads to BSM closed form formula.
- Includes adjustments for dividends and yields through simple adjustments.
- Foundation of modern derivatives pricing.

The Black–Scholes–Merton (BSM) Model

- BSM call price:

$$c_{BSM} = \exp(-r_f\tilde{T})\left[S_t \exp(r_f\tilde{T})\Phi(d) - X\Phi\left(d - \sigma\sqrt{\tilde{T}}\right)\right]$$
$$= S_t\Phi(d) - \exp(-r_f\tilde{T})X\Phi\left(d - \sigma\sqrt{\tilde{T}}\right)$$

Where,

$$d = \frac{\ln(S_t/X) + \tilde{T}\left(r_f + \sigma^2/2\right)}{\sigma\sqrt{\tilde{T}}}$$

- $\Phi(d - \sigma\sqrt{\tilde{T}})$ is the risk-neutral probability of exercise.
- $X\Phi(d - \sigma\sqrt{\tilde{T}})$ is the expected risk-neutral payout when exercising.
- $S_t\Phi(d)\exp(r_f\tilde{T})$ is the risk-neutral expected value of the stock acquired through exercise of the option.
- $\Phi(d)$ measures the sensitivity of the option price to changes in the underlying asset price, S_t, and is referred to as the delta of the option, where $\delta_{BSM} \equiv \frac{\partial c_{BSM}}{\partial S_t}$ is the first derivative of the option with respect to the underlying asset price. This and other sensitivity measures are discussed in detail in the next chapter.

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- **d measures how far the stock price is above or below the strike, adjusted for volatility and time.**
  - If **d is large and positive** → stock is very likely to finish above strike
  - If **d is negative** → less likely
  - If **d = 0** → 50% chance under risk neutrality

- What is risk neutral probability? (Book)
  - **Adjusted so that all assets grow at the risk-free rate**
  - Adjusted probability used for pricing, so expected return = risk-free rate

The Black–Scholes–Merton (BSM) Model

- BSM Put price:

$$p_{BSM} = c_{BSM} + X\exp(-r_f\tilde{T}) - S_t$$
$$= e^{-r_f\tilde{T}}\left\{X\left[1-\Phi\left(d-\sigma\sqrt{\tilde{T}}\right)\right] - S_t\left[1-\Phi(d)\right]e^{r_f\tilde{T}}\right\}$$
$$= e^{-r_f\tilde{T}}X\Phi\left(\sigma\sqrt{\tilde{T}}-d\right) - S_t\Phi(-d)$$

- Where,

$$d = \frac{\ln(S_t/X) + \tilde{T}\left(r_f - q + \sigma^2/2\right)}{\sigma\sqrt{\tilde{T}}}$$

- q is the **Dividend yield (per day)**
- Dividend yield q is the **annual continuous dividend rate** paid by the stock.

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- A **dividend** is a **payment made by a company to its shareholders**, usually from its profits.
- Dividend yield is the annual continuous dividend rate.

If the annual dividend yield = q_{annual}

then **daily dividend yield** is:

$$q_{daily} = \frac{q_{annual}}{365}$$

(assuming 365 days; sometimes 252 trading days are used depending on the model) Suppose:

Annual dividend yield = 4%

$$q_{daily} = \frac{0.04}{365} \approx 0.0001096$$

The stock effectively pays **0.01096% dividend per day** on a continuous basis in the model.

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- **Why is this used in option pricing?**
- In models like **Black–Scholes**, the dividend yield reduces the expected growth of the stock price (under risk-neutral measure)
- Stocks that pay high dividends grow slower in price.
- So call option values fall when dividend yield increases.
- Put option values rise with higher dividend yield.

Numerical Example: Pricing S&P 500 Call

Use the **Black-Scholes pricing model** to price a European call option written on the S&P 500 index.

On January 6, 2010, the **value of the index was 1137.14**. The European call option has a **strike price of 1110** and **43 days to maturity**. The **risk-free interest rate for a 43-day** holding period is found from the T-bill rates to be **0.0006824% per day** (that is, 0.000006824) and the dividend accruing to the index over the next 43 days is expected to be **0.0056967%** per day. For now, we assume the **volatility** of the index is **0.979940%** per day. Thus, we have

$$S_t = 1137.14$$
$$X = 1110$$
$$\tilde{T} = 43$$
$$r_f = 0.0006824\%$$
$$q = 0.0056967\%$$
$$\sigma = 0.979940\%$$

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$$d = \frac{\ln(S_t/X) + \tilde{T}\left(r_f - q + \sigma^2/2\right)}{\sigma\sqrt{\tilde{T}}} = 0.374497, \text{ and } d - \sigma\sqrt{\tilde{T}} = 0.310238$$

which gives

$$\Phi(d) = 0.645983, \text{ and } \Phi\left(d - \sigma\sqrt{\tilde{T}}\right) = 0.621810$$

from which we can calculate the BSM call option price as

$$c_{BSM} = S_t \exp(-q\tilde{T})\Phi(d) - \exp(-r_f\tilde{T})X\Phi\left(d - \sigma\sqrt{\tilde{T}}\right) = 42.77$$

- Paying 42.77 today gives you the opportunity to buy the stock later at a fixed price.
- If the stock rises well above the strike, your payoff can be much higher.
- If it falls below the strike, your maximum loss is the 42.77 paid for the option.

Model Implementation

The simple BSM model implies that a European option price can be written as a non-linear function of six variables,

$$c_{BSM} = c(S_t, r_f, X, \tilde{T}, q; \sigma)$$

- All other inputs are observable directly from markets.
- Volatility is not directly observable.
- Volatility must be estimated—every other variable is known.
- Estimating Volatility by Minimizing Pricing Errors (Page- 233)

$$MSE_{BSM} = \min_{\sigma} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(c_i^{mkt} - c_{BSM}(S_t, r_f, X_i, \tilde{T}_i, q; \sigma) \right)^2 \right\}$$

Why this approach works well: Uses market option prices → captures **market expectations**.

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- Implied Volatility:

$$\sigma_{BSM}^{iv} = c_{BSM}^{-1}\left(S_t, r_f, X, \tilde{T}, q, c^{mkt}\right)$$

Returning to the preceding numerical example of the S&P 500 call option traded on January 6, 2010, knowing that the actual market price for the option was 42.53, we can calculate the implied volatility to be

$$\sigma_{BSM}^{iv} = c_{BSM}^{-1}\left(S_t, r_f, X, \tilde{T}, q, 42.53\right) = 0.971427\%$$

Why We Need GC Instead of BSM

The Black–Scholes–Merton (BSM) model assumes:

- returns are normally distributed
- therefore log prices are lognormal

But real market returns have:

- negative skewness (crashes happen more than big up-moves)
- positive kurtosis (fat tails)
- volatility smile / smirk patterns

➢ Because BSM cannot capture these effects, it misprices options — especially short-maturity and in-the-money options.

➢ **The Gram–Charlier (GC) expansion model fixes this by adding skewness and kurtosis into the pricing formula.**

The Gram–Charlier (GC) Expansion Model

In real financial markets:

- Asset returns are not perfectly normal
- They often show fat tails (large jumps)
- They show skewness (one side heavier)

- The Gram–Charlier model improves this by adjusting the normal distribution to match real-world return patterns.

- GC is an adjusted normal distribution that includes skewness and kurtosis.
- Useful for modeling non-normal returns in finance.
- Helps explain volatility smile and improve option pricing.

Definitions

- Daily log return:

$$R_{t+1} = \ln(S_{t+1}) - \ln(S_t)$$

- Mean Return:

$$E(R_{t+1}) = \mu - \tfrac{1}{2}\sigma^2$$

- Variance:

$$E\left(R_{t+1} - \mu + \tfrac{1}{2}\sigma^2\right)^2 = \sigma^2$$

Skewness and Kurtosis

- Skewness of the one-day return:

$$\zeta_{11} = \frac{E\left(R_{t+1} - \mu + \frac{1}{2}\sigma^2\right)^3}{\sigma^3}$$

- Skewness is informative about the degree of asymmetry of the distribution. A negative skewness arises from large negative returns being observed more frequently than large positive returns.

Skewness and Kurtosis

- Kurtosis of the one-day return:

$$\zeta_{21} = \frac{E\left(R_{t+1} - \mu + \frac{1}{2}\sigma^2\right)^4}{\sigma^4} - 3$$

- Assuming that returns are independent over time, the skewness at horizon $\tilde{T}$ can be written as a simple function of the daily skewness and for kurtosis:

$$\zeta_{1\tilde{T}} = \zeta_{11}/\sqrt{\tilde{T}} \qquad \zeta_{2\tilde{T}} = \zeta_{21}/\tilde{T}$$

- It collapses to the standard normal density when skewness and kurtosis are both zero. Thus GC collapses back to BSM.

GC

- The standardized return at the $\tilde{T}$ -day horizon as

$$w_{\tilde{T}} = \frac{R_{t+1:t+\tilde{T}} - \tilde{T}\left(\mu - \frac{1}{2}\sigma^2\right)}{\sqrt{\tilde{T}}\sigma}$$

$$R_{t+1:t+\tilde{T}} = \left(\mu - \frac{1}{2}\sigma^2\right)\tilde{T} + \sigma\sqrt{\tilde{T}}w_{\tilde{T}}$$

- The distribution is modeled by the **Gram–Charlier expansion**.
- Assume that the standardized returns follow the distribution given by the Gram- Charlier expansion, which is written as

$$f\left(w_{\tilde{T}}\right) = \phi\left(w_{\tilde{T}}\right) - \zeta_{1\tilde{T}}\frac{1}{3!}D^3\phi\left(w_{\tilde{T}}\right) + \zeta_{2\tilde{T}}\frac{1}{4!}D^4\phi\left(w_{\tilde{T}}\right)$$

GC

- GC Call Option Price:

$$c_{GC} \approx S_t \Phi(d) - Xe^{-r_f \tilde{T}} \Phi\left(d - \sqrt{\tilde{T}}\sigma\right)$$

- Estimate them by minimizing:

$$MSE_{GC} = \min_{\sigma, \zeta_{11}, \zeta_{21}} \left\{ \frac{1}{n} \sum_{i=1}^{n} \left(c_i^{mkt} - c_{GC}(S_t, r_f, X_i, \tilde{T}_i; \sigma, \zeta_{11}, \zeta_{21}) \right)^2 \right\}$$

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➢Implied Volatility:

$$\sigma_{GC}^{iv} = c_{BSM}^{-1}\left(S_t, r_f, X, \tilde{T}, c_{GC}\right) \approx \sigma\left[1 - \frac{\zeta_{11}/\sqrt{\tilde{T}}}{3!}d - \frac{\zeta_{21}/\tilde{T}}{4!}\left(1 - d^2\right)\right]$$

GARCH-Based Option Pricing Models

- BSM assumes:
  - constant volatility
  - normally distributed returns
  - no volatility clustering
- But real financial data show:
  - volatility clusters
  - periods of calm then turbulence
  - fat-tailed returns
  - asymmetric volatility (leverage effect)
- GARCH models capture all these features.

How GARCH is Used in Option Pricing

➢The return process:

$$R_{t+1} \equiv \ln(S_{t+1}) - \ln(S_t) = r_f + \lambda\sigma_{t+1} - \frac{1}{2}\sigma_{t+1}^2 + \sigma_{t+1} z_{t+1}$$

with $z_{t+1} \sim N(0,1)$, and $\sigma_{t+1}^2 = \omega + \alpha\left(\sigma_t z_t - \theta\sigma_t\right)^2 + \beta\sigma_t^2$

➢Conditional Expectation and Variance:

$$E_t\left[R_{t+1}\right] = r_f + \lambda\sigma_{t+1} - \frac{1}{2}\sigma_{t+1}^2$$

$$V_t\left[R_{t+1}\right] = \sigma_{t+1}^2$$

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- Show that: $E_t[S_{t+1}/S_t] = \exp(r_f + \lambda\sigma_{t+1})$
- Expected rate of return on the risky asset equals the risk-free rate: $E_t^*[S_{t+1}/S_t] = \exp(r_f)$
- The conditional variance under the risk-neutral process equal: $V_t^*[R_{t+1}] = \sigma_{t+1}^2$

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- Option Price:

$$c_{GH} = \exp(-r_f\tilde{T})E_t^*\left[Max\left\{S_{t+\tilde{T}} - X, 0\right\}\right]$$

$$\approx \exp(-r_f\tilde{T})\frac{1}{MC}\sum_{i=1}^{MC} Max\left\{\check{S}^*_{i,t+\tilde{T}} - X, 0\right\}$$

- This expectation must usually be evaluated via:
  - **Monte Carlo simulation**, or
  - **Numerical integration**

Implied Volatility Function (IVF) Approach

- The Implied Volatility Function (IVF) approach offers a way to price options by modeling the entire implied volatility surface directly, instead of modeling the underlying price dynamics first.
  - Instead of building a model for how the underlying asset price behaves over time,
  - we directly model how the market's implied volatility changes across strikes and maturities and then use that to price options.
- How option prices can be well captured? Discuss the steps. [Page: 244]

Thank You