Definition

In a factorial experimental design, researchers concurrently change and examine a number of factors to see how their cumulative influence affects a dependent variable.

Example

In a study examining the effects of **exercise** (with two levels: low and high) and **diet** (with two levels: low-fat and high fat) on weight loss, a 2x2 factorial design would involve the following conditions:

- Low exercise & low-fat diet
- Low exercise & high-fat diet
- High exercise & low-fat diet
- High exercise & high-fat diet

The researchers recruit participants and randomly assign them to one of these four conditions. They then measure the participants' alertness levels (the dependent variable) after a specific period.

By using this factorial design, the researchers can assess not only the main effects of caffeine intake and exercise on alertness but also the interaction effect between the two factors. For instance, they can determine if high caffeine intake has a different impact on alertness depending on whether a person engages in high or low exercise, or if exercise affects alertness differently in the presence of high or low caffeine intake.

This approach allows researchers to gain a more comprehensive understanding of how these two factors interact and influence alertness levels, which might not be apparent in a single-variable study.

Advantages of Factorial Experiments

- Factorial experiments are more efficient compared to single factor experiments because the main effects as well as the interaction can be estimated in factorial experiments.
- Since hypotheses regarding the main effects and their interactions can be tested separately in factorial experiments, these are more logical.

Disadvantage of Factorial Experiments

- The only disadvantage of factorial experiments is that the number of treatment combinations increase rapidly with the increase of number of factors, or the levels of different factors and analysis of variance components become complicated.

Asymmetrical factorial design
Symmetrical factorial design

2^2 Factorial Experiment

A factorial experiment involving 2 factors each at 2 levels is called 2^2 factorial experiment. Such an experiment has $2^2 = 4$ possible treatment combinations which can be written in standard order as shown below

(1)	a	b	ab

These combinations sometimes expressed as-

(1)	a	b	ab
00	10	01	11

Effect of A at low level of B= a-1

Effect of A at high level of B= ab-b

Total effect of A is [A]=a-1+ab-b

Since these two effects are the simple effect of A at different levels of B. Thus, the main effect of A is: $\frac{1}{2}$ {(ab-b) + (a-1)} = $\frac{1}{2}$(a-1) (b+1)

Effect of AB: $\frac{1}{2}$ {(ab-b) - (a-1)} = $\frac{1}{2}$(a-1) (b-1)

The **interaction effect** (AB)(AB)(AB) measures **how the effect of A changes across the levels of B** (or vice versa).

For CRD with r replicates or RBD with r blocks

The main effect of A is: $\frac{1}{2r}$ {(ab-b) + (a-1)} = $\frac{1}{2r}$(a-1) (b+1)

Effect of AB: $\frac{1}{2r}$ {(ab-b) - (a-1)} = $\frac{1}{2r}$(a-1) (b-1)

Factorial effect	Treatment Combination			
	(1)	a	b	ab
M	+	+	+	+
A	-	+	-	+
B	-	-	+	+
AB	+	-	-	+

$$[A] = ab - b + a - 1$$
$$[B] = ab + b - a - 1$$
$$[AB] = ab - b - a + 1$$

Main Effect of

$$A = \frac{[A]}{2^{2-1}r} = \frac{[A]}{2r}$$
$$\mathrm{B} = \frac{[\mathrm{B}]}{2^{2-1}r} = \frac{[\mathrm{B}]}{2r}$$

Interaction effect

$$= \frac{[\mathrm{AB}]}{2^{2-1}r} = \frac{[\mathrm{AB}]}{2r}$$

Sum of Squares

SS(A)= $\frac{[A]^2}{2^2r} = \frac{[A]^2}{4r}$

SS(B)= $\frac{[B]^2}{2^2r} = \frac{[B]^2}{4r}$

SS(AB)= $\frac{[AB]^2}{2^2r} = \frac{[AB]^2}{4r}$

Hence for 4 treatment we have 3 factorial effects which are mutually orthogonal contrasts of treatment totals.

Treatment SS=SS(A)+SS(B)+SS(AB)

Analysis of data in CRD

Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \qquad i = 0,1; j = 0,1; k = 1(1)r$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)

ANOVA Table

SV	df	SS	MS	F
Treatment	3	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
Error	2^2(r-1)	S_E^2	s_E^2	
Total	2^2r-1	S_{Total}^2	s_{Total}^2	

Empirical problem 1

An experiment was conducted to see the effect of urea (N) and phosphate (P) fertilizers on the yield of a certain variety of rice: N and P were both at 2 levels (0 and 1).

Level of phosphate	Level of Nitrogen			
	0		1	
0	24	24	32	28
	25	30	30	31
1	46	36	30	36
	35	39	30	32

Find main effect and interaction effect by using sign table

Factorial effect	Treatment combination				Effect Total = $\sum sign \times$ yield	Effect $= \frac{\text{Effect Total}}{2*replication}$	SS $= \frac{\text{Effect Total}^2}{2^2*replication}$
	1 (103)	n (121)	p (156)	np (128)			
1	1	1	1	1	508	63.5	
N	-1	1	-1	1	-10	-1.25	6.25
P	-1	-1	1	1	60	7.5	225
NP	1	-1	-1	1	-46	-5.75	132.25

SSTotal=$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{4} y_{ijk}^2 - \frac{y_{...}^2}{16} = 24^2 + 24^2 + \cdots + 32^2 - \frac{(24+24+32)^2}{16} = 16624 - \frac{508^2}{16} = 495$

$SSE = SST - SSN - SSP - SSNP = 131.5$

ANOVA table

SV	df	SS	MS	F
Treatment	3	363.5	121.17	11.06
N	1	6.25	6.25	0.57
P	1	225	225	20.53
NP	1	132.25	132.25	12.07
Error	12	131.5	10.96	
Total	15	495		

Analysis of data in RBD

Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \delta_k + e_{ijk} \qquad i = 0, 1; j = 0,1; k = 1(1)r$

SS(Block)=$\frac{\sum_{k=1}^{r} B_k^2}{2^2} - CT$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2 r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)-SS(Block)

ANOVA Table

SV	df	SS	MS	F
Block	**r-1**	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
Treatment	3	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
Error	$(2^2 - 1)$(r-1)	S_E^2	s_E^2	
Total	2^2r-1	S_{Total}^2	s_{Total}^2	

Empirical problem 2

An experiment was conducted to see the effect of urea (N) and phosphate (P) fertilizers on the yield of a certain variety of rice: N and P were bo th at 2 levels (0 and 1).

	Block							
	I		**II**		**III**		**IV**	
Level of phosphate	**Level of Nitrogen**							
	n_0	n_1	n_0	n_1	n_0	n_1	n_0	n_1
p_0	24	32	24	28	25	30	30	31
p_1	46	30	36	36	35	30	39	32
Total	132		124		120		132	

Find the main effect and interaction effect by using sign table

Factorial effect	**Treatment combination**				**Effect Total =** $\sum \boldsymbol{sign} \times$ **yield**	**Effect** $= \frac{\textbf{Effect Total}}{2*replication}$	**SS** $= \frac{\textbf{Effect Total}^2}{2^2*replication}$
	1 (103)	**n (121)**	**p (156)**	**np (128)**			
1	1	1	1	1	508	63.5	
N	-1	1	-1	1	-10	-1.25	6.25
P	-1	-1	1	1	60	7.5	225
NP	1	-1	-1	1	-46	-5.75	132.25

SSTotal=$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{4} y_{ijk}^2 - \frac{y_{\cdots}^2}{16} = 24^2 + 24^2 + \cdots + 32^2 - \frac{(24+24+32)^2}{16} = 16624 - \frac{508^2}{16} = 495$

SS(Block)=$\frac{\sum_{k=1}^{r} B_k^2}{2^2} - CT = \frac{132^2+124^2+120^2+132^2}{4} - \frac{(132+124+120+132)^2}{2^2\times 4} = 27$

$SSE = SST - SSN - SSP - SSNP - SSBlock = 104.5$

ANOVA table

SV	**df**	**SS**	**MS**	**F**
Block	3	27	9	0.78
Treatment	3	363.5	121.17	10.43
N	1	6.25	6.25	0.54
P	1	225	225	19.38
NP	1	132.25	132.25	11.39
Error	9	104.5	11.61	
Total	15	495		

Yates Method

Treatment Combination	Yield	Column 1	Column 2	Main effect & interaction	SS
M	[1]	[1] + [a] = w	w + x = GT		
A	[a]	[b] + [ab] = x	y + z = [A]	$\frac{[A]}{2r}$	$\frac{[A]^2}{4r}$
B	[b]	[a] - [1] = y	x – w = [B]	$\frac{[B]}{2r}$	$\frac{[B]^2}{4r}$
AB	[ab]	[ab] - [b] = z	z – y =[AB]	$\frac{[AB]}{2r}$	$\frac{[AB]^2}{4r}$

Analysis of data in CRD

Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk} \quad i = 0, 1; j = 0,1; k = 1(1)r$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2 r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)

ANOVA Table

SV	df	SS	MS	F
Treatment	3	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
Error	2^2(r-1)	S_E^2	s_E^2	
Total	2^2r-1	S_{Total}^2	s_{Total}^2	

Empirical problem 3

An experiment was conducted to see the effect of urea (N) and phosphate (P) fertilizers on the yield of a certain variety of rice: N and P were both at 2 levels (0 and 1).

Level of phosphate	Level of Nitrogen			
	0		1	
0	24	24	32	28
	25	30	30	31
1	46	36	30	36
	35	39	30	32

Find main effect and interaction effect by using yates method. Construct ANOVA table.

Yates's method

Treatment combination	Total Yield	Column 1	Column 2	Effect $= \frac{\text{Effect Total}}{2 * replication}$	SS $= \frac{\text{Effect Total}^2}{2^2 * replication}$
1	103	224	508	63.5	
n	121	284	-10	-1.25	6.25
p	156	18	60	7.5	225
np	128	-28	-46	-5.75	132.25

SSTotal=$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{4} y_{ijk}^2 - \frac{y_{...}^2}{16} = 24^2 + 24^2 + \cdots + 32^2 - \frac{(24+24+32)^2}{16} = 16624 - \frac{508^2}{16} = 495$

$SSE = SST - SSN - SSP - SSNP = 131.5$

ANOVA table

SV	df	SS	MS	F
Treatment	3	363.5	121.17	11.06
N	1	6.25	6.25	0.57
P	1	225	225	20.53
NP	1	132.25	132.25	12.07
Error	12	131.5	10.96	
Total	15	495		

Analysis of data in RBD

Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \delta_k + e_{ijk} \qquad i = 0,1; j = 0,1; k = 1(1)r$

SS(Block)=$\frac{\sum_{k=1}^{r} B_k^2}{2^2} - CT$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2 r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)-SS(Block)

ANOVA Table

SV	df	SS	MS	F
Block	**r-1**	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
Treatment	3	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
Error	(2^2-1)(r-1)	S_E^2	s_E^2	
Total	2^2r-1	S_{Total}^2	s_{Total}^2	

Empirical problem 4

An experiment was conducted to see the effect of urea (N) and phosphate (P) fertilizers on the yield of a certain variety of rice: N and P were both at 2 levels (0 and 1).

	Block							
	I		**II**		**III**		**IV**	
Level of phosphate	**Level of Nitrogen**							
	n_0	n_1	n_0	n_1	n_0	n_1	n_0	n_1
p_0	24	32	24	28	25	30	30	31
p_1	46	30	36	36	35	30	39	32
Total	132		124		120		132	

Find the main effect and interaction effect by using yates' method. Construct ANOVA table.

Yates's method

Treatment combination	**Total Yield**	**Column 1**	**Column 2**	**Effect** $=\frac{\text{Effect Total}}{2*replication}$	**SS** $=\frac{\text{Effect Total}^2}{2^2*replication}$
1	103	224	508	63.5	
n	121	284	-10	-1.25	6.25
p	156	18	60	7.5	225
np	128	-28	-46	-5.75	132.25

SSTotal=$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{4} y_{ijk}^2 - \frac{y_{\dots}^2}{16} = 24^2 + 24^2 + \cdots + 32^2 - \frac{(24+24+32)^2}{16} = 16624 - \frac{508^2}{16} = 495$

SS(Block)=$\frac{\sum_{k=1}^{r} B_k^2}{2^2} - CT = \frac{132^2+124^2+120^2+132^2}{4} - \frac{(132+124+120+132)^2}{2^2\times 4} = 27$

$SSE = SST - SSN - SSP - SSNP - SSBlock = 104.5$

ANOVA table

SV	**df**	**SS**	**MS**	**F**
Block	3	27	9	0.78
Treatment	3	363.5	121.17	10.43
N	1	6.25	6.25	0.54
P	1	225	225	19.38
NP	1	132.25	132.25	11.39
Error	9	104.5	11.61	
Total	15	495		

Linear combination

(1)	A	B	ab
00	10	01	11

Let us define a variable

$$x_i = \begin{cases} 0 & ; \quad if \;\; i^{th} \;\; treatment \;\; absent \\ 1 & ; \quad if \;\; i^{th} \;\; treatment \;\; present \end{cases}$$

Now, for treatment A we can write the following pair of linear equation.

$$\left. \begin{aligned} 1.x_1 + 0.x_2 &= 0 \\ &= 1 \end{aligned} \right\} \text{mod } 2$$

$$A_o = 00 + 01$$
$$A_1 = 10 + 11$$
$$[A] = A_1 - A_o$$

Effect of A =$\frac{[A]}{2^{2-1}r}$

SS(A)= $\frac{A_o^2 + A_1^2}{2^{2-1}r} - CT$; or SS(A)= $\frac{[A]^2}{2^2 r}$

Now, for treatment B we can write the following pair of linear equation.

$$\left. \begin{aligned} 0.x_1 + 1.x_1 &= 0 \\ &= 1 \end{aligned} \right\} \text{mod } 2$$

$$B_o = 00 + 10$$
$$B_1 = 01 + 11$$
$$[B] = B_1 - B_o$$

Effect of B $=\frac{[B]}{2^{2-1}r}$

SS(B)= $\frac{B_o^2+B_1^2}{2^{2-1}r} - CT$; or SS(B)= $\frac{[B]^2}{2^2 r}$

Now, for treatment AB we can write the following pair of linear equation.

$$1.x_1 + 1.x_1 \begin{array}{l} = 0 \\ = 1 \end{array}\Bigg\} \text{mod } 2$$

$$AB_o = 00 + 11$$
$$AB_1 = 01 + 10$$
$$[AB] = AB_1 - AB_o$$

Effect of AB $=\frac{[AB]}{2^{2-1}r}$

SS(B)= $\frac{AB_o^2+AB_1^2}{2^{2-1}r} - CT$; or SS(AB)= $\frac{[AB]^2}{2^2 r}$

Analysis of data in CRD

Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + e_{ijk}$ $\quad i = 0,1; j = 0,1; k = 1(1)r$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2 r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)

ANOVA Table

SV	df	SS	MS	F
Treatment	3	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
Error	2^2(r-1)	S_E^2	s_E^2	
Total	2^2r-1	S_{Total}^2	s_{Total}^2	

Empirical problem 3

An experiment was conducted to see the effect of urea (N) and phosphate (P) fertilizers on the yield of a certain variety of rice: N and P were both at 2 levels (0 and 1).

<table>
<tr><th rowspan="2">Level of phosphate</th><th colspan="4">Level of Nitrogen</th></tr>
<tr><td colspan="2">0</td><td colspan="2">1</td></tr>
<tr><td rowspan="2">0</td><td>24</td><td>24</td><td>32</td><td>28</td></tr>
<tr><td>25</td><td>30</td><td>30</td><td>31</td></tr>
<tr><td rowspan="2">1</td><td>46</td><td>36</td><td>30</td><td>36</td></tr>
<tr><td>35</td><td>39</td><td>30</td><td>32</td></tr>
</table>

Find main effect and interaction effect by using the concept of linear combination approach Construct ANOVA table.

$$A_o = 00 + 01 = 24 + 24 + 25 + 30 + 32 + 28 + 30 + 31 = 224$$
$$A_1 = 10 + 11 = 46 + 36 + 35 + 39 + 30 + 36 + 30 + 32 = 284$$
$$[A] = A_1 - A_o = 284 - 224 = 60$$

Effect of A $=\frac{[A]}{2^{2-1}r} = \frac{60}{2\times4} = 7.5$

SS(A)= $\frac{A_o^2+A_1^2}{2^{2-1}r} - CT = \frac{224^2+284^2}{2\times4} - \frac{508^2}{16} = 225$ or SS(A)= $\frac{[A]^2}{2^2r} = \frac{60^2}{4\times4} = 225$

Similarly, obtain

Effect of B $=\frac{[B]}{2^{2-1}r}$

SS(B)= $\frac{B_o^2+B_1^2}{2^{2-1}r} - CT$; or SS(B)= $\frac{[B]^2}{2^2r}$

SSTotal=$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{4} y_{ijk}^2 - \frac{y_{\dots}^2}{16} = 24^2 + 24^2 + \cdots + 32^2 - \frac{(24+24+32)^2}{16} = 16624 - \frac{508^2}{16} = 495$

$SSE = SST - SSN - SSP - SSNP = 131.5$

ANOVA table

SV	df	SS	MS	F
Treatment	3	363.5	121.17	11.06
N	1	6.25	6.25	0.57
P	1	225	225	20.53
NP	1	132.25	132.25	12.07
Error	12	131.5	10.96	
Total	15	495		

Analysis of data in RBD

Model

$Y_{ijk} = \mu + \alpha_i + \beta_j + (\alpha\beta)_{ij} + \delta_k + e_{ijk} \quad i = 0, 1; j = 0,1; k = 1(1)r$

SS(Block)=$\frac{\sum_{k=1}^{r} B_k^2}{2^2} - CT$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)-SS(Block)

ANOVA Table

SV	**df**	**SS**	**MS**	**F**
Block	**r-1**	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
Treatment	3	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
Error	(2^2-1)(r-1)	S_E^2	s_E^2	
Total	2^2r-1	S_{Total}^2	s_{Total}^2	

Empirical problem 6

An experiment was conducted to see the effect of urea (N) and phosphate (P) fertilizers on the yield of a certain variety of rice: N and P were both at 2 levels (0 and 1).

	Block							
	I		**II**		**III**		**IV**	
Level of phosphate	**Level of Nitrogen**							
	n_0	n_1	n_0	n_1	n_0	n_1	n_0	n_1
p_0	24	32	24	28	25	30	30	31
p_1	46	30	36	36	35	30	39	32
Total	132		124		120		132	

$$A_o = 00 + 01 = 24 + 24 + 25 + 30 + 32 + 28 + 30 + 31 = 224$$
$$A_1 = 10 + 11 = 46 + 36 + 35 + 39 + 30 + 36 + 30 + 32 = 284$$
$$[A] = A_1 - A_o = 284 - 224 = 60$$

Effect of A $=\frac{[A]}{2^{2-1}r} = \frac{60}{2\times4} = 7.5$

SS(A)= $\frac{A_0^2+A_1^2}{2^{2-1}r} - CT = \frac{224^2+284^2}{2\times4} - \frac{508^2}{16} = 225$ or SS(A)= $\frac{[A]^2}{2^2 r} = \frac{60^2}{4\times4} = 225$

Similarly, obtain

Effect of B $=\frac{[B]}{2^{2-1}r}$

SS(B)= $\frac{B_0^2+B_1^2}{2^{2-1}r}-CT$;   or SS(B)= $\frac{[B]^2}{2^2 r}$

SSTotal=$\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{4}y_{ijk}^2-\frac{y_{...}^2}{16}=24^2+24^2+\cdots+32^2-\frac{(24+24+32)^2}{16}=16624-\frac{508^2}{16}=495$

SS(Block)=$\frac{\sum_{k=1}^{r}B_k^2}{2^2}-CT=\frac{132^2+124^2+120^2+132^2}{4}-\frac{(132+124+120+132)^2}{2^2\times 4}=27$

$SSE=SST-SSN-SSP-SSNP-SSBlock=104.5$

ANOVA table

SV	df	SS	MS	F
Block	3	27	9	0.78
Treatment	3	363.5	121.17	10.43
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NP	1	132.25	132.25	11.39
Error	9	104.5	11.61	
Total	15	495		

2^3 Factorial Experiment

A factorial experiment involving 3 factors each at 2 levels is called 2^3 factorial experiment. Such an experiment has $2^2=4$ possible treatment combinations which can be written in standard order as shown below

(1) a b ab c ac bc abc

These combinations sometimes expressed as-

(1)	a	b	ab	c	ac	bc	abc
000	100	010	110	001	101	011	111

Now the effect of A, B, C are evaluated as follows;

Simple effect of A at low level of B and low level of C =a-(1)

Simple effect of A at high level of B and low level of C=ab-b

Simple effect of A at low level of B and high level of C=ac-c

Simple effect of A at high level of B and high level of C=abc-bc

Total effect of A is [A]= abc-bc+ ac-c+ ab-b+ a-(1) =(a-1) (b+1) (c+1)

Main effect of A is= $\frac{1}{4}$ (abc-bc+ ac-c+ ab-b+ a-(1)) = $\frac{1}{4}$(a-1) (b+1) (c+1)

Similarly,

Main effect of B is=$\frac{1}{4}$(a+1) (b-1) (c+1)

Main effect of C is=$\frac{1}{4}$(a+1) (b+1) (c-1)

The effect of AB indicates the effect of A in presence/highlevel of B minus the effect of A in absence/low level of B

Effect of A at high level of B= $\mathrm{abc} - \mathrm{bc} + \mathrm{ab} - \mathrm{b}$

Effect of A at low level of B= $\mathrm{ac} - \mathrm{c} + \mathrm{a} - (1)$

Total Interaction effect of AB is [AB]= $\mathrm{abc} - \mathrm{bc} + \mathrm{ab} - \mathrm{b} - (\mathrm{ac} - \mathrm{c}) - \left(\mathrm{a} - (1)\right) = (a-1)(b-1)(c+1)$

Interaction effect of AB is $= \frac{1}{4}\{\mathrm{abc} - \mathrm{bc} + \mathrm{ab} - \mathrm{b} - (\mathrm{ac} - \mathrm{c}) - \left(\mathrm{a} - (1)\right)\} = \frac{1}{4}(a-1)(b-1)(c+1)$

Similarly, Interaction effect of BC is $= \frac{1}{4}(a+1)(b-1)(c-1)$

Interaction effect of AC is $= \frac{1}{4}(a-1)(b+1)(c-1)$

Total effect of ABC is $[ABC]$ = abc-bc-(ac-c)- (ab-b)+ a-(1)= $(a-1)(b-1)(c-1)$

Interaction effect of ABC is $= \frac{1}{4}(a-1)(b+1)(c-1)$

For CRD with r replicates or RBD with r blocks

The main effect of A is=$\frac{1}{4r}(a+1)(b-1)(c-1)$

Interaction Effect of AB= $\frac{1}{4r}(a-1)(b-1)(c+1)$

ABC=$\frac{1}{4r}(a-1)(b-1)(c-1)$

Factorial effect	Treatment Combination							
	(1)	*a*	*b*	*ab*	c	ac	bc	abc
M	+	+	+	+	+	+	+	+
A	-	+	-	+	-	+	-	+
B	-	-	+	+	-	-	+	+
AB	+	-	-	+	+	-	-	+
C	-	-	-	-	+	+	+	+
AC	+	-	+	-	-	+	-	+
BC	+	+	-	-	-	-	+	+
ABC	-	+	+	-	+	-	-	+

$$[A] = abc - bc + ac - c + ab - b + a - 1$$
$$[B] = abc + bc - ac - c + ab + b - a - 1$$
$$[c] = abc + bc + ac + c - ab - b - a - 1$$
$$[AB] = abc - bc - ac + c + ab - b - a + 1$$
$$[BC] = abc + bc - ac - c - ab - b + a + 1$$
$$[AC] = abc - bc + ac - c - ab + b - a + 1$$
$$[ABC] = abc - bc - ac + c - ab + b + a - 1$$

Main Effect of

$$A = \frac{[A]}{2^{3-1}r} = \frac{[A]}{4r}$$
$$\mathrm{B} = \frac{[\mathrm{B}]}{2^{3-1}r} = \frac{[\mathrm{B}]}{4r}$$
$$C = \frac{[\mathrm{B}]}{2^{3-1}r} = \frac{[\mathrm{B}]}{4r}$$

Interaction effect

$$AB = \frac{[\mathrm{AB}]}{2^{3-1}r} = \frac{[\mathrm{AB}]}{4r}$$
$$BC = \frac{[\mathrm{BC}]}{2^{3-1}r} = \frac{[\mathrm{BC}]}{4r}$$
$$AC = \frac{[\mathrm{AC}]}{2^{3-1}r} = \frac{[\mathrm{AC}]}{4r}$$
$$ABC = \frac{[\mathrm{ABC}]}{2^{3-1}r} = \frac{[\mathrm{ABC}]}{4r}$$

Sum of Squares

SS(A)= $\frac{[A]^2}{2^3r} = \frac{[A]^2}{8r}$

SS(B)= $\frac{[B]^2}{2^3r} = \frac{[B]^2}{8r}$

SS(C)= $\frac{[C]^2}{2^3r} = \frac{[C]^2}{8r}$

SS(AB)= $\frac{[AB]^2}{2^3r} = \frac{[AB]^2}{8r}$

SS(BC)= $\frac{[BC]^2}{2^3r} = \frac{[BC]^2}{8r}$

SS(AC)= $\frac{[AC]^2}{2^3r} = \frac{[AC]^2}{8r}$

SS(ABC)= $\frac{[ABC]^2}{2^3r} = \frac{[ABC]^2}{8r}$

Hence for 8 treatment we have 7 factorial effects which are mutually orthogonal contrasts of treatment totals.

Treatment SS=SS(A)+SS(B)+SS(C)+SS(AB)+SS(BC)+SS(AC)+SS(ABC)

Hence treatment SS has been partitioned into 7 orthogonal components each carrying $1\ d.f.$

Yates Method

Treatment Combination	**Yield**	**Column 1**	**Column 2**	**Column 2**	Main effect & interaction	**SS**
(1)	[1]	[1] + [a] = x_1	$x_1 + x_2 = y_1$	$y_1 + y_2 = GT$		
a	[a]	[b] + [ab] = x_2	$x_3 + x_4 = y_2$	$y_3 + y_4 = [A]$	$\frac{[A]}{4r}$	$\frac{[A]^2}{8r}$
b	[b]	[c]+[ac]= x_3	$x_5 + x_6 = y_3$	$y_5 + y_6 = [B]$	$\frac{[B]}{4r}$	$\frac{[B]^2}{8r}$
ab	[ab]	[bc]+[abc]= x_4	$x_7 + x_8 = y_4$	$y_7 + y_8 = [AB]$	$\frac{[AB]}{4r}$	$\frac{[AB]^2}{8r}$
c	[c]	[a] - [1] = x_5	$x_2 - x_1 = y_5$	$y_2 - y_1 = [C]$	$\frac{[C]}{4r}$	$\frac{[C]^2}{8r}$
ac	[ac]	[ab] - [b] = x_6	$x_4 - x_3 = y_6$	$y_4 - y_3 = [AC]$	$\frac{[AC]}{4r}$	$\frac{[AC]^2}{8r}$
bc	[bc]	$[ac] - [c] = x_7$	$x_6 - x_5 = y_6$	$y_6 - y_5 = [BC]$	$\frac{[BC]}{4r}$	$\frac{[BC]^2}{8r}$
abc	[abc]	$[abc] - [bc] = x_7$	$x_8 - x_7 = y_8$	$y_8 - y_7 = [ABC]$	$\frac{[ABC]}{4r}$	$\frac{[ABC]^2}{8r}$

Linear combination

(1)	a	b	ab	c	ac	bc	abc
000	100	010	110	001	101	011	111

Let us define a variable

$$x_i = \begin{cases} 0 & ; \quad if \quad i^{th} \quad treatment \quad absent \\ 1 & ; \quad if \quad i^{th} \quad treatment \quad present \end{cases}$$

Now, for treatment A we can write the following pair of linear equation.

$$1.x_1 + 0.x_2 + 0.x_3 \left.\begin{matrix} = 0 \\ = 1 \end{matrix}\right\} \text{mod } 2$$

$$A_o = 000 + 010 + 001 + 011$$

$A_1 = 100 + 110$+101+111

$$[A] = A_1 - A_o$$

Effect of A =$\frac{[A]}{2^{3-1}r}$

SS(A)= $\frac{A_0^2+A_1^2}{2^{3-1}r} - CT$;   or SS(A)= $\frac{[A]^2}{2^3r}$

Now, for treatment B we can write the following pair of linear equation.

$$\left.\begin{aligned} 0.x_1 + 1.x_2 + 0.x_3 &= 0 \\ &= 1 \end{aligned}\right\} \text{mod}$$

$$B_o = 000 + 100 + 001 + 101$$

$$B_1 = 010 + 110 + 011 + 111$$

$$[B] = B_1 - B_o$$

Effect of B $=\frac{[B]}{2^{3-1}r}$

SS(B)$= \frac{B_o^2+B_1^2}{2^{3-1}r} - CT$; or SS(B)$= \frac{[B]^2}{2^3 r}$

Now, for treatment C we can write the following pair of linear equation.

$$\left.\begin{aligned} 0.x_1 + 0.x_2 + 1.x_3 &= 0 \\ &= 1 \end{aligned}\right\} \text{mod}$$

$$C_o = 000 + 100 + 010 + 110$$

$$C_1 = 001 + 101 + 011 + 111$$

$$[C] = C_1 - C_o$$

Effect of C $=\frac{[C]}{2^{3-1}r}$

SS(C)$= \frac{C_o^2+C_1^2}{2^{3-1}r} - CT$; or SS(C)$= \frac{[C]^2}{2^3 r}$

Now, for treatment AB we can write the following pair of linear equation.

$$\left.\begin{aligned} 1.x_1 + 1.x_2 + 0.x_3 &= 0 \\ &= 1 \end{aligned}\right\} \text{mod } 2$$

$$AB_o = 000 + 001 + 110 + 111$$

$AB_1 = 100 + 010$+101+011

$$[AB] = AB_0 - AB_1$$

Effect of AB $=\frac{[AB]}{2^{3-1}r}$

SS(AB)$= \frac{AB_o^2+AB_1^2}{2^{3-1}r} - CT$; or SS(AB)$= \frac{[AB]^2}{2^3 r}$

Now, for treatment BC we can write the following pair of linear equation.

$$\left.\begin{aligned} 0.x_1 + 1.x_2 + 1.x_3 &= 0 \\ &= 1 \end{aligned}\right\} \text{mod } 2$$

$$BC_o = 000 + 011 + 100 + 111$$

$BC_1 = 010 + 001$+101+110

$$[BC] = BC_0 - BC_1$$

Effect of BC $=\frac{[BC]}{2^{3-1}r}$

SS $(BC)$$= \frac{BC_o^2+BC_1^2}{2^{3-1}r} - CT$; or SS $(BC)$$= \frac{[BC]^2}{2^3 r}$

Now, for treatment AC we can write the following pair of linear equation.

$$1.x_1 + 0.x_2 + 1.x_3 \begin{rcases} = 0 \\ = 1 \end{rcases} \text{mod } 2$$

$$AC_o = 000 + 010 + 101 + 111$$

$AC_1 = 100 + 001$+110+011

$$[AC] = AC_0 - AC_1$$

Effect of AC =$\frac{[AC]}{2^{3-1}r}$

SS (AC)= $\frac{AC_0^2+AC_1^2}{2^{3-1}r} - CT$; or SS ($AC$)= $\frac{[AC]^2}{2^3 r}$

Now, for treatment ABC we can write the following pair of linear equation.

$$1.x_1 + 1.x_2 + 1.x_3 \begin{rcases} = 0 \\ = 1 \end{rcases} \text{mod } 2$$

$$ABC_o = 000 + 011 + 110 + 101$$

$ABC_1 = 100 + 010$+001+111

$$[ABC] = ABC_1 - ABC_0$$

Effect of ABC =$\frac{[ABC]}{2^{3-1}r}$

SS (ABC)= $\frac{ABC_0^2+ABC_1^2}{2^{3-1}r} - CT$; or SS ($ABC$)= $\frac{[ABC]^2}{2^3 r}$

Analysis of data in CRD

Model

$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_K + (\alpha\beta)_{ij} + (\beta\gamma)_{jk} + (\alpha\gamma)_{ik} + (\alpha\beta\gamma)_{ijk} + e_{ijkl}$ $i = 0,1; j =$ 0,1; $k = 0,1$; l=1(1)r

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^2 r}$

SSE= SS (Total)- Treatment SS= SS (Total)- SS(A)-SS(B)-SS(AB)-SS(BC)-SS(AC)-SS(ABC)

ANOVA Table

SV	**df**	**SS**	**MS**	**F**
Treatment	7	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$

AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$
C	1	S_C^2	s_C^2	$\frac{s_C^2}{s_E^2}$
AC	1	S_{AC}^2	s_{AC}^2	$\frac{s_{AC}^2}{s_E^2}$
BC	1	S_{BC}^2	s_{BC}^2	$\frac{s_{BC}^2}{s_E^2}$
ABC	1	S_{ABC}^2	s_{ABC}^2	$\frac{s_{ABC}^2}{s_E^2}$
Error	2^3(r-1)	S_E^2	s_E^2	
Total	2^3r-1	S_{Total}^2	s_{Total}^2	

Analysis of data in RBD

Model

$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_K + (\alpha\beta)_{ij} + (\beta\gamma)_{jk} + (\alpha\gamma)_{ik} + (\alpha\beta\gamma)_{ijk} + \delta_k + e_{ijkl}$ $\quad i = 0, 1; j = 0,1; k = 0,1;$ l=1(1)r

SS(Block)=$\frac{\sum_{k=1}^{r} B_k^2}{2^2} - CT$

SS (Total) = $\sum_{i=0}^{1}\sum_{j=0}^{1}\sum_{k=1}^{r} y_{ijk}^2 - CT$; CT=$\frac{GT}{2^3 r}$

SSE= SS (Total)- Treatment SS= SS (Total)-SS(A)-SS(B)-SS(AB)-SS(BC)-SS(AC)-SS(ABC)

ANOVA Table

SV	**df**	**SS**	**MS**	**F**
Block	r-1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
Treatment	7	S_T^2	s_T^2	$\frac{s_T^2}{s_E^2}$
A	1	S_A^2	s_A^2	$\frac{s_A^2}{s_E^2}$
B	1	S_B^2	s_B^2	$\frac{s_B^2}{s_E^2}$
AB	1	S_{AB}^2	s_{AB}^2	$\frac{s_{AB}^2}{s_E^2}$

C	1	S_C^2	s_C^2	$\frac{s_C^2}{s_E^2}$
AC	1	S_{AC}^2	s_{AC}^2	$\frac{s_{AC}^2}{s_E^2}$
BC	1	S_{BC}^2	s_{BC}^2	$\frac{s_{BC}^2}{s_E^2}$
ABC	1	S_{ABC}^2	s_{ABC}^2	$\frac{s_{ABC}^2}{s_E^2}$
Error	$(2^3 - 1)$(r-1)	S_E^2	s_E^2	
Total	2^3r-1	S_{Total}^2	s_{Total}^2	