9-1.3 The 3^3 Design

Now suppose there are three factors (A, B, and C) under study, and each factor is at three levels arranged in a factorial experiment. This is a 3^3 factorial design, and the experimental layout and treatment combination notation were shown previously in Figure 9-2. The 27 treatment combinations have 26 degrees of freedom. Each main effect has 2 degrees of freedom, each two-factor interaction has 4 degrees of freedom, and the three-factor interaction has 8 degrees of freedom. If there are n replicates, there are $n3^3 - 1$ total degrees of freedom and $3^3(n - 1)$ degrees of freedom for error.

The sums of squares may be calculated using the standard methods for factorial designs. In addition, if the factors are quantitative the main effects may be partitioned into linear and quadratic components, each with a single degree of freedom. The two-factor interactions may be decomposed into linear × linear, linear × quadratic, quadratic × linear, and quadratic × quadratic effects. Finally, the three-factor interaction ABC can be partitioned into eight single-degree-of-freedom components corresponding to linear × linear × linear, linear × linear × quadratic, and so on. Such a breakdown for the three-factor interaction is generally not very useful.

It is also possible to partition the two-factor interactions into their I and J components. These would be designated AB, AB^2, AC, AC^2, BC, and BC^2, and each component would have two degrees of freedom. As in the 3^2 design, these components have no physical significance.

The three-factor interaction ABC may be partitioned into four orthogonal two-degrees-of-freedom components, which are usually called the W, X, Y, and Z components of the interaction. They are also referred to as the AB^2C^2, AB^2C, ABC^2, and ABC components of the ABC interaction, respectively. The two notations are used interchangeably; that is,

$$W(ABC) = AB^2C^2$$
$$X(ABC) = AB^2C$$
$$Y(ABC) = ABC^2$$
$$Z(ABC) = ABC$$

Note that no first letter can have an exponent other than 1. Like the I and J components, the W, X, Y, and Z components have no practical interpretation. They are, however, useful in constructing more complex designs.

EXAMPLE 9-1 ....

A machine is used to fill 5-gallon metal containers with soft drink syrup. The variable of interest is the amount of syrup loss due to frothing. Three factors are thought to influence frothing: the nozzle design (A), the filling speed (B), and the operating pressure (C). Three nozzles, three filling speeds, and three pressures are chosen and two replicates of a 3^3 factorial experiment are run. The coded data are shown in Table 9-1 on page 368.

The analysis of variance for the syrup loss data is shown in Table 9-2 on page 368. The sums of squares have been computed by the usual methods. We see that the filling speed and operating pressure are statistically significant. All three two-factor interactions are also significant. The two-factor interactions are analyzed graphically in Figure 9-4 on page 369. The middle level of speed gives the best performance, nozzle types 2 and 3, and either the low (10 psi) or high (20 psi) pressure seem most effective in reducing syrup loss.

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Table 9-1 Syrup Loss Data for Example 9-1 (units are cubic centimeters − 70)

	Nozzle Type (A)								
	1			2			3		
	Speed (in RPM) (B)								
Pressure (in psi) (C)	100	120	140	100	120	140	100	120	140
10	−35	−45	−40	17	−65	20	−39	−55	15
	−25	−60	15	24	−58	4	−35	−67	−30
15	110	−10	80	55	−55	110	90	−28	110
	75	30	54	120	−44	44	113	−26	135
20	4	−40	31	−23	−64	−20	−30	−61	54
	5	−30	36	−5	−62	−31	−55	−52	4

Example 9-1 illustrates a situation where the three-level design often finds some application; one or more of the factors is **qualitative**, naturally taking on three levels, and the remaining factors are **quantitative**. In this example, suppose that there are only three nozzle designs that are of interest. This is clearly, then, a qualitative factor that requires three levels. The filling speed and the operating pressure are quantitative factors. Therefore, we could fit a quadratic model such as Equation 9-1 in the two factors speed and pressure at each level of the nozzle factor.

Table 9-3 (on the facing page) shows these quadratic regression models. The β's in these models were estimated using a standard linear regression computer program. (We will discuss least squares regression in more detail in Chapter 10.) In these models, the variables x_1 and x_2 are coded to the levels -1, 0, $+1$ as discussed previously, and we assumed the following natural levels for pressure and speed:

Coded Level	Speed (psi)	Pressure (rpm)
−1	100	10
0	120	15
+1	140	20

Table 9-3 presents models both in terms of these coded variables and in terms of the natural levels of speed and pressure.

Table 9-2 Analysis of Variance for Syrup Loss Data

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	*P*-Value
A, nozzle	993.77	2	496.89	1.17	0.3256
B, speed	61,190.33	2	30,595.17	71.74	<0.0001
C, pressure	69,105.33	2	34,552.67	81.01	<0.0001
AB	6,300.90	4	1,575.22	3.69	0.0383
AC	7,513.90	4	1,878.47	4.40	0.0222
BC	12,854.34	4	3,213.58	7.53	0.0025
ABC	4,628.76	8	578.60	1.36	0.2737
Error	11,515.50	27	426.50		
Total	174,102.83	53			

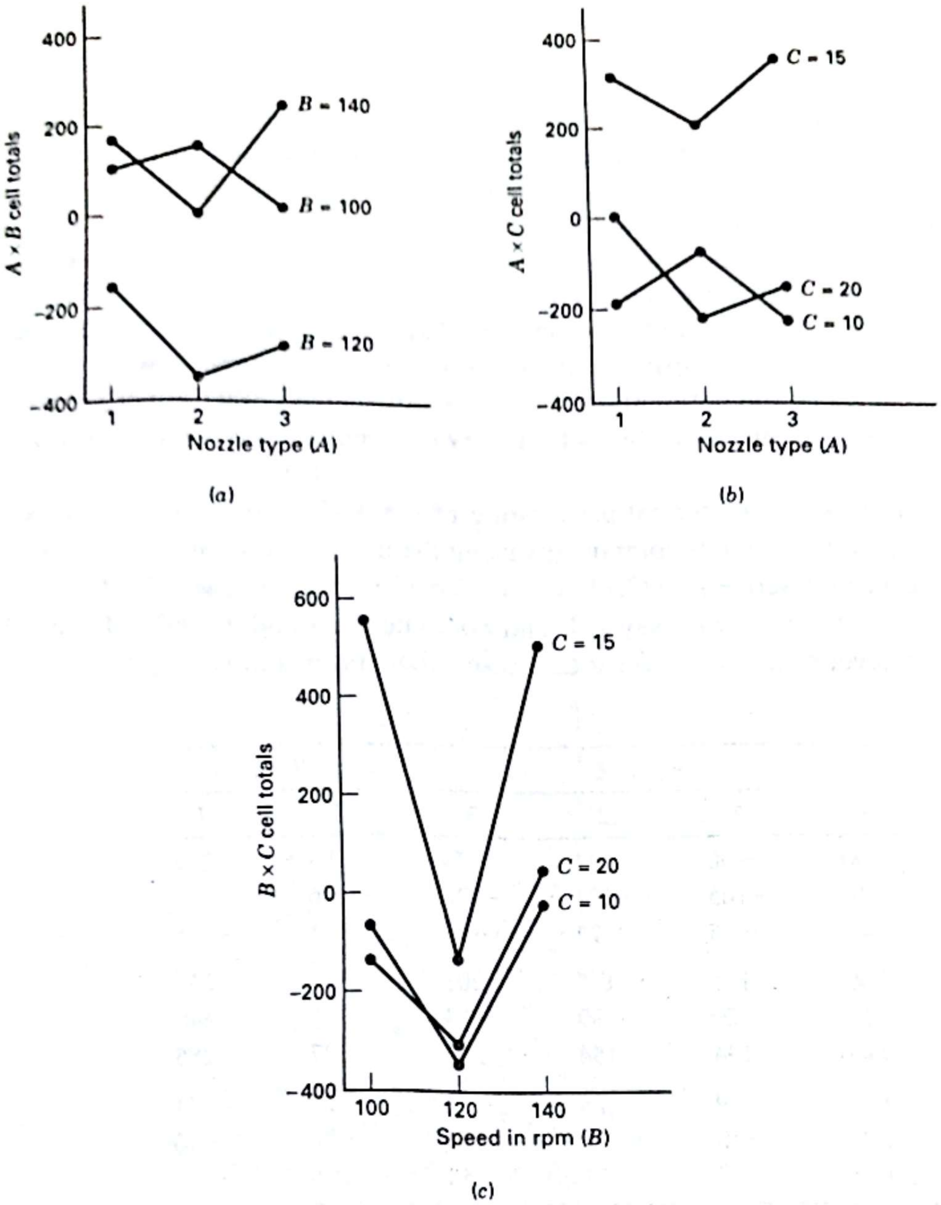

Figure 9-4 Two-factor interactions for Example 9-1.

Table 9-3 Regression Models for Example 9-1

Nozzle Type	x_1 = Speed (S), x_2 = Pressure (P) in Coded Units
1	$\hat{y} = 22.1 + 3.5x_1 + 16.3x_2 + 51.7x_1^2 - 71.8x_2^2 + 2.9x_1x_2$ $\hat{y} = 1217.3 - 31.256S + 86.017P + 0.12917S^2 - 2.8733P^2 + 0.02875SP$
2	$\hat{y} = 25.6 - 22.8x_1 - 12.3x_2 + 14.1x_1^2 - 56.9x_2^2 - 0.7x_1x_2$ $\hat{y} = 180.1 - 9.475S + 66.75P + 0.035S^2 - 2.2767P^2 - 0.0075SP$
3	$\hat{y} = 15.1 + 20.3x_1 + 5.9x_2 + 75.8x_1^2 - 94.9x_2^2 + 10.5x_1x_2$ $\hat{y} = 1940.1 - 40.058S + 102.48P + 0.18958S^2 - 3.7967P^2 + 0.105SP$