3^2 Factorial Experiment

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A factorial experiment with two factors A *and* B each at three levels $0,\ 1,\ 2$ is called 3^2 factorial experiment. Such a factorial experiment has $3^2 = 9$ possible treatment combinations which are shown in the following table.

		B			Total
		b_0 0	b_1 1	b_2 2	
A	a_0 > 0	a_0b_0 00	a_0b_1 01	a_0b_2 02	A_0
	a_1 > 1	a_1b_0 10	a_1b_1 11	a_1b_2 12	A_1
	a_2 > 2	a_2b_0 20	a_2b_1 21	a_2b_2 22	A_2
Total		B_0	B_1	B_2	G

Effect of Different Factor

The effect of factor A can be determined as follows-

The change in the yield due to changing the level of A from 0 to 1 level is

$$\begin{aligned} A_1 - A_0 &= a_1b_0 + a_1b_1 + a_1b_2 - a_0b_0 - a_0b_1 - a_0b_2 \\ &= b_0\left(a_1 - a_0\right) + b_1\left(a_1 - a_0\right) + b_2\left(a_1 - a_0\right) \\ &= \left(a_1 - a_0\right)\left(b_0 + b_1 + b_2\right) \end{aligned}$$

Similarly, change in the yield due to changing the level of A from 1 to 2 is-

$$\begin{aligned} A_2 - A_1 &= a_2b_0 + a_2b_1 + a_2b_2 - a_1b_0 - a_1b_1 - a_1b_2 \\ &= (a_2 - a_1)(b_0 + b_1 + b_2) \end{aligned}$$

The main effect of A can be partitioned into linear effect and quadratic effect.

Now, adding the above two change and dividing by r (replication) we get the linear effect of A and can be written as-

$$A' = \frac{1}{r}(a_2 - a_0)(b_0 + b_1 + b_2)$$

The quadratic effect of A can be obtained as-

$$\begin{aligned} A'' &= \frac{1}{2r}(A_2 - A_1 - A_1 + A_0) \\ &= \frac{1}{2r}(a_2 - 2a_1 + a_0)(b_0 + b_1 + b_2) \end{aligned}$$

Similarly,

$$B' = \frac{1}{r}(b_2 - b_0)(a_0 + a_1 + a_2)$$

$$B'' = \frac{1}{2r}(b_2 - 2b_1 + b_0)(a_0 + a_1 + a_2)$$

The linear effect and quadratic effect both are independent and orthogonal and are estimated with 1 $d.f$.

Now, the interaction effect AB (with 4 $d.f.$) can be partitioned into four components i.e. into 4 contrast (each with 1 $d.f.$). These 4 contrast can be obtained by multiplication of linear and quadratic effect.

$$A'B' = \frac{1}{2r}(a_2 - a_0)(b_2 - b_0) \qquad with\ 1\ d.f.$$

$$A''B' = \frac{1}{4r}(a_2 - 2a_1 + a_0)(b_2 - b_0) \qquad with\ 1\ d.f.$$

$$A'B'' = \frac{1}{4r}(a_2 - a_0)(b_2 - 2b_1 + b_0) \qquad with\ 1\ d.f.$$

$$A''B'' = \frac{1}{8r}(a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0) \qquad with\ 1\ d.f.$$

Yates' Method for Computing Factorial Effects and Their Respective Sum of Square

Treatment Combination	Total Yield	Column-1	Column-2	Main & interaction effect	Divisor	SS
00	x_1	$x_1+x_2+x_3=y_1$	$y_1+y_2+y_3=G$	G	$9r$	
10	x_2	$x_4+x_5+x_6=y_2$	$y_4+y_5+y_6=[A']$	A'	$6r$	$SS(A')=\frac{[A']^2}{6r}$
20	x_3	$x_7+x_8+x_9=y_3$	$y_7+y_8+y_9=[A'']$	A''	$18r$	$SS(A'')=\frac{[A'']^2}{18r}$
01	x_4	$x_3-x_1=y_4$	$y_3-y_1=[B']$	B'	$6r$	$SS(B')=\frac{[B']^2}{6r}$
11	x_5	$x_6-x_4=y_5$	$y_6-y_4=[A'B']$	$A'B'$	$4r$	$SS(A'B')=\frac{[A'B']^2}{4r}$
21	x_6	$x_9-x_7=y_6$	$y_9-y_7=[A''B']$	$A''B'$	$12r$	$SS(A''B')=\frac{[A''B']^2}{12r}$
02	x_7	$x_3-2x_2+x_1=y_7$	$y_3-2y_2+y_1=[B'']$	B''	$18r$	$SS(B'')=\frac{[B'']^2}{18r}$
12	x_8	$x_6-2x_5-x_4=y_8$	$y_6-2y_5+y_4=[A'B'']$	$A'B''$	$12r$	$SS(A'B'')=\frac{[A'B'']^2}{12r}$
22	x_9	$x_9-2x_8-x_7=y_9$	$y_9-2y_8+y_7=[A''B'']$	$A''B''$	$36r$	$SS(A''B'')=\frac{[A''B'']^2}{36r}$

In Yates' method, we compute different SS with 1 $d.f$.

Computation of SS Using Linear Equation

The treatment combination of 3^2 factorial experiment can be written as-

00 01 02 10 11 12 20 21 22

For the main effect A:

The linear equation will be-

$$\left.\begin{aligned} 1\cdot x_1 + 0\cdot x_2 &= 0 \\ &= 1 \\ &= 2 \end{aligned}\right\} \bmod 3$$

$$\therefore \quad \begin{aligned} A_0 &= 00 + 01 + 02 \\ A_1 &= 10 + 11 + 12 \\ A_2 &= 20 + 21 + 22 \end{aligned}$$

$$\therefore \quad SS(A) = \frac{A_0^2 + A_1^2 + A_2^2}{3r} - C.T. \qquad with\ 2\ d.f.$$

$$Similarly, \quad SS(B) = \frac{B_0^2 + B_1^2 + B_2^2}{3r} - C.T. \qquad with\ 2\ d.f.$$

The interaction effect AB (with $4\ d.f.$) can be split into two components, $AB\ and\ AB_2$ (each with $2\ d.f.$).

Now for AB linear equation will be-

$$\left.\begin{aligned} 1\cdot x_1 + 1\cdot x_2 &= 0 \\ &= 1 \\ &= 2 \end{aligned}\right\} \bmod 3$$

$$\therefore \quad \begin{aligned} (AB)_0 &= 00 + 12 + 21 \\ (AB)_1 &= 01 + 10 + 22 \\ (AB)_2 &= 20 + 02 + 11 \end{aligned}$$

$$\therefore \quad SS(AB) = \frac{(AB)_0^2 + (AB)_1^2 + (AB)_2^2}{3r} - C.T. \qquad with\ 2\ d.f.$$

For AB_2 the linear equation will be-

$$\left.\begin{aligned} 1\cdot x_1 + 2\cdot x_2 &= 0 \\ &= 1 \\ &= 2 \end{aligned}\right\} \bmod 3$$

$$\therefore \quad \begin{aligned} (AB_2)_0 &= 00 + 11 + 22 \\ (AB_2)_1 &= 10 + 02 + 21 \\ (AB_2)_2 &= 20 + 12 + 01 \end{aligned}$$

$$\therefore \quad SS(AB_2)=\frac{(AB_2)_0^2+(AB_2)_1^2+(AB_2)_2^2}{3r}-C.T. \qquad with\ 2\ d.f.$$

For analysis of data in RBD

$$Total(SS)=\sum_{i=1}^{r}\sum_{j=1}^{3^2} y_{ij}^2-C.T. \qquad where,\ C.T.=\frac{G^2}{3^2 r}$$

$$Bolck(SS)=\frac{\sum_{i=1}^{r} R_i^2}{3^2}-C.T.$$

$$Treatment(SS)=\frac{\sum_{j=1}^{3^2} T_j^2}{r}-C.T.=SS(A)+SS(B)+SS(AB)+SS(AB_2)$$

$$Error(SS)=Total(SS)-Block(SS)-Treatment(SS)$$

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ANOVA-Table for 3^2 factorial experiment

Source of Variation	$d.f.$	SS	MSS	F
Replicate	$r-1$	S_R^2	$s_b^2=S_b^2/r-1$	
Treatments	$3^2-1=8$	S_t^2	$s_t^2=S_t^2/8$	$\frac{s_t^2}{s_e^2}=F_t$
A	2	S_A^2	$s_A^2=S_A^2/2$	$\frac{s_A^2}{s_e^2}=F_A$
B	2	S_B^2	$s_B^2=S_B^2/2$	$\frac{s_B^2}{s_e^2}=F_B$
AB	2	S_{AB}^2	$s_{AB}^2=S_{AB}^2/2$	$\frac{s_{AB}^2}{s_e^2}=F_{AB}$
AB_2	2	$S_{AB_2}^2$	$s_{AB_2}^2=S_{AB_2}^2/2$	$\frac{s_{AB_2}^2}{s_e^2}=F_{AB_2}$
Error	$(3^2-1)(r-1)$	S_e^2	$s_e^2=S_e^2/(3^2-1)(r-1)$	
Total	$3^2 r-1$			

Inference Procedure

The hypothesis is

H_0 : All the main effect and interaction effect are insignificant

against H_1 : All the main effect and interaction effect are significant

Test Statistic, $F = \frac{MS\ \ of\ \ a\ \ component}{Error\ \ MS}$ which follows F -distribution with 2 and $(3^2-1)(r-1)$ $d.f$

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Decision Rule

For a particular treatment, if $F_{cal} > F_{\alpha,\, tab}$ then we reject H_0 at $a\%$ level of significance otherwise we accept it.