3^3 Factorial Experiment

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A factorial experiment with three factors $A,\ B\ and\ C$ each at three levels $0,\ 1,\ 2$ is called 3^3 factorial experiment. Such a factorial experiment has $3^3=27$ possible treatment combinations which are shown in the following table.

000	100	200	010	110	210	020	120	220
001	101	201	011	111	211	021	121	221
002	102	202	012	112	212	022	122	222

The linear and quadratic effects of the factor of 3^3 factorial experiment are estimated with $1\ d.f.$ and are obtained as follows-

$$A'=\frac{1}{9r}(a_2-a_0)(b_0+b_1+b_2)(c_0+c_1+c_2)$$
$$A''=\frac{1}{18r}(a_2-2a_1+a_0)(b_0+b_1+b_2)(c_0+c_1+c_2)$$
$$B'=\frac{1}{9r}(a_0+a_1+a_2)(b_2-b_0)(c_0+c_1+c_2)$$
$$A'B'=\frac{1}{6r}(a_2-a_0)(b_2-b_0)(c_0+c_1+c_2)$$
$$A''B'=\frac{1}{12r}(a_2-2a_1+a_0)(b_2-b_0)(c_0+c_1+c_2)$$
$$B''=\frac{1}{18r}(a_0+a_1+a_0)(b_2-2b_1+b_0)(c_0+c_1+c_2)$$
$$A'B''=\frac{1}{12r}(a_2-a_0)(b_2-2b_1+b_0)(c_0+c_1+c_2)$$
$$A''B''=\frac{1}{24r}(a_2-2a_1+a_0)(b_2-2b_1+b_0)(c_0+c_1+c_2)$$
$$C'=\frac{1}{9r}(a_0+a_1+a_2)(b_0+b_1+b_2)(c_2-c_2)$$
$$A'C'=\frac{1}{6r}(a_2-a_0)(b_0+b_1+b_2)(c_2-c_0)$$
$$A''C'=\frac{1}{12r}(a_2-2a_1+a_0)(b_0+b_1+b_2)(c_2-c_0)$$
$$B'C'=\frac{1}{6r}(a_0+a_1+a_2)(b_2-b_0)(c_2-c_0)$$

$$B''C' = \frac{1}{12r}(a_0 + a_1 + a_2)(b_0 + b_1 + b_2)(c_2 - c_0)$$
$$C'' = \frac{1}{18r}(a_0 + a_1 + a_2)(b_0 + b_1 + b_2)(c_2 - 2c_1 + c_0)$$
$$A'C'' = \frac{1}{12r}(a_2 - a_0)(b_0 + b_1 + b_2)(c_2 - 2c_1 + c_0)$$
$$A''C'' = \frac{1}{24r}(a_2 - 2a_1 + a_0)(b_0 + b_1 + b_2)(c_2 - 2c_1 + c_0)$$
$$B'C'' = \frac{1}{12r}(a_0 + a_1 + a_2)(b_2 - b_0)(c_2 - 2c_1 + c_0)$$
$$B''C'' = \frac{1}{24r}(a_0 + a_1 + a_2)(b_2 - 2b_1 + b_0)(c_2 - 2c_1 + c_0)$$
$$A'B'C' = \frac{1}{4r}(a_2 - a_0)(b_2 - b_0)(c_2 - c_0)$$
$$A'B'C'' = \frac{1}{8r}(a_2 - a_0)(b_2 - b_0)(c_2 - 2c_1 + c_0)$$
$$A'B''C' = \frac{1}{8r}(a_2 - a_0)(b_2 - 2b_1 + b_0)(c_2 - c_0)$$
$$A'B''C'' = \frac{1}{16r}(a_2 - a_0)(b_2 - 2b_1 + b_0)(c_2 - 2c_1 + c_0)$$
$$A''B'C' = \frac{1}{8r}(a_2 - 2a_1 + a_0)(b_2 - b_0)(c_2 - c_0)$$
$$A''B'C'' = \frac{1}{16r}(a_2 - 2a_1 + a_0)(b_2 - b_0)(c_2 - 2c_1 + c_0)$$
$$A''B''C' = \frac{1}{16r}(a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0)(c_2 - c_0)$$
$$A''B''C'' = \frac{1}{32r}(a_2 - 2a_1 + a_0)(b_2 - 2b_1 + b_0)(c_2 - 2c_1 + c_0)$$

Computation of Factor and Interaction SS Using Linear Equation

The treatment combination of 3^3 factorial experiment involving three factors $A,\ B\ and\ C$ each at three levels $0,\ 1\ and\ 2$ can written as-

000	010	020	001	011	020	002	012	022
100	110	120	101	111	120	102	112	122
200	210	220	201	211	220	202	212	222

For treatment A the linear equation will be-

$$\left.\begin{aligned} 1\cdot x_1+0\cdot x_2+0\cdot x_3 &= 0\\ &= 1\\ &= 2 \end{aligned}\right\} \bmod\ 3$$

$$\therefore \quad \begin{aligned} A_0 &= 000+010+020+001+011+021+002+012+022\\ A_1 &= 100+110+120+101+111+121+102+112+122\\ A_2 &= 200+210+220+201+211+221+202+212+222 \end{aligned}$$

$$\therefore \quad SS(A)=\frac{A_0^2+A_1^2+A_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$Similarly, \quad SS(B)=\frac{B_0^2+B_1^2+B_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(C)=\frac{C_0^2+C_1^2+C_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

Now the Interaction term AB can be split into two component AB *and* AB_2 (each with two $d.f.$).

Now the linear equation for AB and AB_2 will be-

$$\left.\begin{aligned} 1\cdot x_1+1\cdot x_2+0\cdot x_3 &= 0\\ &= 1\\ &= 2 \end{aligned}\right\} \bmod\ 3 \qquad \left.\begin{aligned} 1\cdot x_1+2\cdot x_2+0\cdot x_3 &= 0\\ &= 1\\ &= 2 \end{aligned}\right\} \bmod\ 3$$

$$\therefore \quad \begin{aligned} (AB)_0 &= 000+120+121+122+210+211+212+001+002\\ (AB)_1 &= 010+011+012+100+101+102+220+221+222\\ (AB)_2 &= 020+021+022+110+111+112+200+201+202 \end{aligned}$$

$$\therefore \quad SS(AB)=\frac{(AB)_0^2+(AB)_1^2+(AB)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

Similarly, $$SS(AB_2)=\frac{(AB_2)_0^2+(AB_2)_1^2+(AB_2)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(AC)=\frac{(AC)_0^2+(AC)_1^2+(AC)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(AC_2)=\frac{(AC_2)_0^2+(AC_2)_1^2+(AC_2)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(BC)=\frac{(BC)_0^2+(BC)_1^2+(BC)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(BC_2)=\frac{(BC_2)_0^2+(BC_2)_1^2+(BC_2)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(ABC)=\frac{(ABC)_0^2+(ABC)_1^2+(ABC)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(ABC_2)=\frac{(ABC_2)_0^2+(ABC_2)_1^2+(ABC_2)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(AB_2C)=\frac{(AB_2C)_0^2+(AB_2C)_1^2+(AB_2C)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

$$SS(AB_2C_2)=\frac{(AB_2C_2)_0^2+(AB_2C_2)_1^2+(AB_2C_2)_2^2}{3^2 r}-C.T. \qquad with\ 2\ d.f.$$

For analysis of data in RBD

$$Total(SS)=\sum_{i=1}^{r}\sum_{j=1}^{3^3} y_{ij}^2-C.T. \qquad where,\ C.T.=\frac{G^2}{3^3 r}$$

$$Bolck(SS)=\frac{\sum_{i=1}^{r} B_i^2}{3^3}-C.T.$$

$$Treatment(SS)=\frac{\sum_{j=1}^{3^3} T_j^2}{r}-C.T.=SS(A)+\cdots+SS(AB)+SS(AB_2)+\cdots+SS(AB_2C_2)$$

$$Error(SS)=Total(SS)-Block(SS)-Treatment(SS)$$

ANOVA-Table for 3^3 factorial experiment

Source of Variation	$d.f.$	SS	MSS	F
Replicate	$r-1$	S_R^2	$s_b^2 = S_b^2 / r-1$	
Treatments	$3^3-1=26$	S_t^2	$s_t^2 = S_t^2 / 26$	$\frac{s_t^2}{s_e^2} = F_t$
A	2	S_A^2	$s_A^2 = S_A^2 / 2$	$\frac{s_A^2}{s_e^2} = F_A$
B	2	S_B^2	$s_B^2 = S_B^2 / 2$	$\frac{s_B^2}{s_e^2} = F_B$
AB	2	S_{AB}^2	$s_{AB}^2 = S_{AB}^2 / 2$	$\frac{s_{AB}^2}{s_e^2} = F_{AB}$
AB_2	2	$S_{AB_2}^2$	$s_{AB_2}^2 = S_{AB_2}^2 / 2$	$\frac{s_{AB_2}^2}{s_e^2} = F_{AB_2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
AB_2C_2	2	$S_{AB_2C_2}^2$	$s_{AB_2C_2}^2 = S_{AB_2C_2}^2 / 2$	$\frac{s_{AB_2C_2}^2}{s_e^2} = F_{AB_2C_2}$
Error	$(3^3-1)(r-1)$	S_e^2	$s_e^2 = S_e^2 / (3^3-1)(r-1)$	
Total	3^3r-1			

Inference Procedure

The hypothesis is

$$H_0: \text{All the main effect and interaction effect are insignificant}$$

$$against \quad H_1: \text{All the main effect and interaction effect are significant}$$

Test Statistic, $F = \frac{MS\ of\ a\ component}{Error\ MS}$ which follows F-distribution with 2 and $(3^3-1)(r-1)$ $d.f$.

Decision Rule

For a particular treatment, if $F_{cal} > F_{\alpha,\, tab}$ then we reject H_0 at $a\%$ level of significance otherwise we accept it.