

Fundamentals of Theory of Interest

Principle: The initial amount of money we invest.

Accumulated value: The total amount of money we have after our investment grows over time, including both the principle and any interest or returns earned.

Amount function: If $t > 0$ is the length of time for which the principle has been invested, then the amount of money at that time will be denoted by $A(t)$, is called amount function.

$$A(t) = k \cdot a(t); \quad k = A(0) \\ \quad \quad \quad \quad \quad \quad \quad \quad = \text{principle}$$

$a(t) = \text{accumulated value}$

Accumulation function: It is denoted as $a(t)$, represents how an initial principle amount grows over time.

$$a(t) = \frac{A(t)}{A(0)}$$

Amount of Interest

Accumulated value - Principle

$$= A(t) - A(0)$$

Properties of Accumulation Function!

- (i) $a(0) = 1$
- (ii) $a(t)$ is generally an increasing function.
- (iii) $a(t)$ is continuous if interest occurs.

Interest

Interest is the additional amount earned on an investment or loan.

Why compensation?

- Interest rate risk
- Default risk

Types of Interest

- 1) Simple interest
- 2) compound interest

Simple Interest:

Simple interest is used occasionally, primarily between integer interest periods. In this situation, the amount of interest earned each year is constant.

In other words, (only the original principal earns interest from year to year) and interest accumulated in any given year does not earn interest in the future years.

$$A(t) = K(1 + it)$$

i = effective rate of interest

K = principle amount

Simple interest can be estimated by following methods:

- 1) Exact simple interest method
- 2) Ordinary simple interest method on Banker's rule.

$$\frac{(1 - n)A - (n)A}{(1 - n)A} = ni$$

- In this method, we use the exact number of days in the numerator and 365 days in the denominator. It is used in Canada.
- Here we use 360 days. It is used in USA.

Compound Interest

Compound interest is interest calculation on both the initial amount (principal) and the interest that has already been added. This means that interest builds on interest, making the total grow faster over time.

$$A(t) = K(1+i)^t$$

Effective rate of Interest:

It shows how much interest is earned if interest is paid at the end of the year.

$$\begin{aligned} i_m &= \frac{\text{Interest earned}}{\text{amount invested}} \\ &= \frac{A(n) - A(n-1)}{A(n-1)} \end{aligned}$$

If the period is one year, the effective rate of interest is also referred to as the effective annual rate of interest.

Simple VS compound Interest

- i) simple and compound interest produce same return over a one-year period.
- ii) compound interest is commonly for ^{long} ~~short~~ term; simple interest is mainly for short term.
- iii) compound interest grows exponentially and simple interest grows linearly.
- ~~ix) Long vs~~
- iv) compound interest is better for terms over a year; simple interest may be better for under a year, i.e.,
 $(1+i)^t > (1+it)$ if $t > 1$
 $(1+i)^t < (1+it)$, $0 < t < 1$

(v) The formula for compound interest $a(t) = (1+i)^t$ and the formula for simple interest is $a(t) = 1+it$; $t > 0$

(vi) Rate of compound interest and rate of effective interest are identical i.e., $i_m = i$, whereas, rate of simple interest and effective rate are not identical i.e.,

$$i_m = \frac{i}{1+i(n-1)}$$

Present value

It represents the current worth of a future amount of money, taking into account interest, since money can earn interest over time, a specific amount today is generally more valuable than the same amount received in the future.

For simple interest,

$$P(1+it) = x$$

$$\Rightarrow P = \frac{x}{1+it}$$

For compound interest,

$$P(1+i)^t = x$$

$$\Rightarrow P = \frac{x}{(1+i)^t} = x v^t$$

Nominal Rate of Interest:

- In many real life situations, the interest period is not a year, but rather some shorter period e.g., six months (semiannual), a month, a week etc.

- The lender tries to keep the fact hidden

- lender's benefit to do,

- The word nominal means "in name only" or so called.

$$A(t) = A(0) \cdot \left[1 + \frac{i(m)}{m} \right]^{mt}$$

If i is the effective rate of interest per year,

$$1 + i = \left[1 + \frac{i(m)}{m} \right]^m$$

$$\Rightarrow 1 + \frac{i(m)}{m} = (1+i)^{1/m}$$

$$\Rightarrow i(m) = m \left[(1+i)^{1/m} - 1 \right]$$

Examples:

(1) Given,

Principal = \$1000

15% interest per year

$$A(2) = K (1+i)^t$$

$$= 1000 (1+0.15)^2$$

$$= \$1322.5$$

$$\therefore \text{Amount of interest} = \$1322.5 - \$1000 = \$322.5$$

(2)

Jack borrowed \$1000

15% interest per year in advance,

future obligation = \$1000

Interest for the year = $P \times 0.15$

$$P + P \times 0.15 = 1000$$

$$\Rightarrow P = \frac{1000}{1.15} = \$869.5652$$

The bank will pay \$869.5652 at present to Jack and he will repay the bank \$1000 after one year,

(3) Given,

$$P = \$1000, = A(0)$$

$$A(2) = \$1322.5$$

$$\text{Interest rate} = \frac{A(2) - A(0)}{A(0)} \times 100$$

$$= \frac{1322.5 - 1000}{1000} \times 100$$

$$= 32.25\%$$

Effective annual rate of interest

$$16.125\%$$

(4) $P = \$200$

$t = 3$

rate of simple interest is 3% per annum

$$\therefore A(B) = P(1 + it)$$

$$= 200(1 + 0.03 \times 3)$$

$$= \$218$$

(5)

In 1990, January 1 the amount of debt is

$$1000(1 + 0.15)^4 = 1749.006$$

On January 17, 1990,

$$1749.006 \left[1 + 0.15 \times \frac{16}{365} \right]$$

$$= 1760.51$$

$$1749.006 \left[1 + 0.15 \times \frac{16}{360} \right]$$

$$= 1760.67$$

(6)

$$P(1+i)^t = 1000$$

$$\Rightarrow P(1+0.10)^3 = 1000$$

$$\therefore P = 751.315 \text{ (present value)}$$

(7) Jack borrows 1000 at a rate of 15% compound interest per year.

(a)

$$A(5) = 1000(1+0.15)^5$$

$$= 2011.357 \text{ Tk}$$

(b) How much he owe after 57 days = ?

$$t = \frac{57}{365}$$

$$A\left(\frac{57}{365}\right) = 1000\left(1+0.15 \times \frac{57}{365}\right)$$

$$= 1023.42 \text{ Tk}$$

(c) 1 year 57 days = ?

$$A(1) = 1000(1+0.15)^1$$

$$= 1150 \text{ Tk}$$

$$1150 \left(1 + 0.15 \times \frac{57}{365} \right)$$

$$= 1176.94 \text{ Tk}$$

(d)

$$1000 (1 + 0.15)^t = 2000$$

$$\Rightarrow (1.15)^t = 2$$

$$\Rightarrow \ln (1.15)^t = \ln 2$$

$$\Rightarrow t \ln (1.15) = \ln 2$$

$$\therefore t = 4.96 \text{ years} \approx 5 \text{ years}$$

(8)

$$i = 8\% = 0.08$$

$$t = 4 \text{ years}$$

(a)

$$x (1 + i)^t = 93500$$

$$\Rightarrow x (1 + 0.08 \times 4) = 93500$$

$$\Rightarrow x = \frac{93500}{1.32}$$

$$\therefore x = 70833.33 \text{ Tk}$$

(b)

$$x(1+i)^t = 93500$$

$$\Rightarrow x(1+0.08)^4 = 93500$$

$$\therefore x = 68725.29 \text{ TK}$$

(9)

$$P = 1000 \text{ TK}$$

Effective rate of interest $i = 24\%$
per month
he over after 3

years = 36

$$A(3) = 1000(1+0.02)^{36}$$

$$= 2039.89$$

(10) Find the accumulated value of 1000 after 3 years at a nominal rate of interest of 24% per year convertible monthly.

We know that,

$$1+i = \left[1 + \frac{i(m)}{m}\right]^m$$

$$\Rightarrow i = \left[1 + \frac{i(m)}{m}\right]^m - 1$$

$$= \left[1 + \frac{0.24}{12}\right]^{12} - 1$$

$$= 0.2682$$

$$\therefore A(3) = 1000 (1 + 0.2682)^3 (d) \\ = 2039.69 \text{ TK}$$

(ii)

Given, $i(6) = 0.15$

$$1+i = \left[1 + \frac{i(6)}{6} \right]^6$$

$$\Rightarrow 1+i = \left[1 + \frac{0.15}{6} \right]^6$$

and $1+i = \left[1 + \frac{i(2)}{2} \right]^2$

$$\left[1 + \frac{i(2)}{2} \right]^2 = \left[1 + \frac{0.15}{6} \right]^6$$

$$\Rightarrow 1 + \frac{i(2)}{2} = (1.025)^3$$

$$\Rightarrow \frac{i(2)}{2} = (1.025)^3 - 1$$

$$\Rightarrow i(2) = 0.15378$$

Assignment

(1) Find the accumulated value of \$200 invested for four years, if the rate of simple interest is 8% per annum?

Soln:

Given,

$$\text{Principal, } P = \$200$$

$$i = 0.08$$

$$t = 4 \text{ years}$$

∴ Accumulated value

$$A(t) = P(1 + it)$$

$$\Rightarrow A(4) = 200(1 + 0.08 \times 4)$$

$$\therefore A(4) = \$264$$

(2) Jack borrows \$2000 from the bank on August 11, 2016 at a rate of 10% interest per year. How much does he owe on August 18, 2017?

Soln:

$$P = \$2000$$

$$i = 0.10$$

In August 11, 2017, the amount of debt is,

$$2000 (1 + 0.10)^1 \\ = \$2200$$

On August 18, 2017,

$$2200 \left[1 + 0.10 \times \frac{7}{365} \right] \\ = \$2204.22$$

(3) A man borrows \$500 at a nominal rate of 15% per year convertible quarterly. How much does he owe after two years?

Soln:

We know that,

$$1 + i = \left[1 + \frac{i^{(m)}}{m} \right]^m$$

$$\Rightarrow i = \left[1 + \frac{i^{(m)}}{m} \right]^m - 1$$

$$\Rightarrow i = \left[1 + \frac{0.15}{4} \right]^4 - 1$$

$$\Rightarrow i = (1.0375)^4 - 1$$

$$\therefore i = 0.15865$$

$\therefore$ Accumulated value,

$$A(2) = 500 (1 + 0.15865)^2$$

$$= \$671.23$$

(4) Mr. Abdullah borrowed \$1000

in a bank at rate of 13% compound interest per year for 3 years.

How much does he get at present?

Also, compute present value considering simple interest.

Soln!

Suppose,

x = present value

$$\therefore x (1+i)^t = 1000$$

$$c) x (1 + 0.13)^3 = 1000$$

$$\Rightarrow x (1.43)^3 = 1000$$

$$\therefore x = \$693.05$$

$$x(1+i)^t = 1000$$

$$\Rightarrow x(1+0.13 \times 3) = 1000$$

$$\Rightarrow 1.39x = 1000$$

$$\therefore x = \$719.42$$

(7) Ebtiaj deposits \$8000 in a savings account at Federal credit Union Bank. For the first three years the money grows at annual effective rate of compound interest 7%, for the next two years it is 4% convertible semiannually, and for the following four years it is 6% convertible semiannually. If Ebtiaj closes his account after nine years, what will his balance be?

Soln:

(1) First 3 years,

$$A(3) = 8000 \times (1 + 0.07)^3 \\ = \$9800.344$$

$$\therefore A_1 = A(3) = \$9800.344$$

(2) for the next two years

$$A_2 = A_1 \cdot \left(1 + \frac{0.04}{2}\right)^{2 \times 2}$$

$$= 9800.344 (1.02)^4$$

$$= 10608.20752$$

(3) for the following four years.

$$A_3 = A_2 \cdot \left(1 + \frac{0.06}{2}\right)^{2 \times 4}$$

$$= 10608.20752 \left(1 + \frac{0.06}{2}\right)^8$$

$$= 13438.1599$$

After nine years, Ebtia's balance will be approximately \$13438.16

Discount:

A discount is a deduction applied at the beginning of a period on the amount due at the end of that period.

Effective rate of discount

$$d = \frac{a(1) - 1}{a(1)}$$

in the m th year,

$$d_m = \frac{a(m) - a(m-1)}{a(m)}$$

The effective rate of discount d is the ratio of the amount of interest earned during the year to the amount invested at the end of the year.

Nominal rate of discount:

$$1 - d = \left[1 - \frac{d(m)}{m} \right]^m$$

$$\Rightarrow (1 - d)^{1/m} = 1 - \frac{d(m)}{m}$$

$$\Rightarrow \frac{d(m)}{m} = 1 - (1 - d)^{1/m}$$

$$\therefore d(m) = m [1 - (1 - d)^{1/m}]$$

Some basic relationships:

(i)

$$d = \frac{ae^{j\omega} - 1}{ae^{j\omega}} = \frac{1 + j - 1}{1 + j - 1} = \frac{j}{1 + j}$$

$$\therefore d = \frac{j}{1 + j}$$

(ii)

$$d = \frac{j}{1 + j} = jv$$

$$\Rightarrow d = jv \quad \text{where } v = (1 + j)^{-1}$$

(iii)

$$d + jd = j$$

$$\Rightarrow j = \frac{d}{1 - d}$$

(iv)

$$d = \frac{j}{1 + j}$$

$$= \frac{j(1 - j)}{(1 + j)(1 - j)} = \frac{j(1 - j)}{1 - j^2}$$

$$= 1 - v$$

$$\therefore d = 1 - v$$

$$\therefore v = 1 - d$$

Relation between $i(m)$ and $d(m)$

We know that

$$1 - d = \left[1 - \frac{d(m)}{m} \right]^m$$

$$\text{since, } 1 - d = \frac{1}{1+i}$$

$$\therefore 1+i = \left[1 - \frac{d(m)}{m} \right]^{-m}$$

$$\Rightarrow \left[1 + \frac{i(m)}{m} \right]^m = \left[1 - \frac{d(m)}{m} \right]^{-m}$$

$$\Rightarrow 1 + \frac{i(m)}{m} = \frac{1}{1 - \frac{d(m)}{m}}$$

$$\Rightarrow 1 - \frac{d(m)}{m} = \frac{1}{1 + \frac{i(m)}{m}}$$

$$\Rightarrow \frac{d(m)}{m} = 1 - \frac{1}{1 + \frac{i(m)}{m}}$$

$$\Rightarrow d(m) = m \left[\frac{\frac{i(m)}{m}}{1 + \frac{i(m)}{m}} \right]$$

$$\therefore d(m) = m \left[\frac{i(m)}{m \left[1 + \frac{i(m)}{m} \right]} \right]$$

$$\therefore d(m) = \frac{i(m)}{1 + \frac{i(m)}{m}}$$

Examples!

(a) $AV = 1000$

$d = 0.09$

$t = 3$

$x = ?$

$$x(1-d)^{-t} = 1000$$

$$\Rightarrow x(1-0.09)^{-3} = 1000$$

$$\Rightarrow x = \frac{1000}{(1-0.09)^{-3}}$$

$$\therefore x = 753.571$$

(b)

$$d = \frac{i}{1+i} - 1$$

$$\Rightarrow d + 1 = \frac{i}{1+i}$$

$$\Rightarrow d = \frac{i}{1+i} - 1$$

$$\Rightarrow d = \frac{i(1+i)}{1+i} - 1$$

$$\Rightarrow i = \frac{d}{1-d} = \frac{0.09}{1-0.09} = 0.0989$$

Also
Assignment-6

(2)

Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of 12% per year convertible monthly.

Soln:

$$\left[1 - \frac{d^{(n)}}{n}\right]^n = \left[1 + \frac{i^{(m)}}{m}\right]^m$$

$$\Rightarrow \left[1 - \frac{d^{(2)}}{2}\right]^2 = \left[1 + \frac{0.12}{12}\right]^{12}$$

$$\Rightarrow 1 - \frac{d^{(2)}}{2} = \left[1 + \frac{0.12}{12}\right]^{-6}$$

$$\Rightarrow \frac{d^{(2)}}{2} = 1 - \left[1 + \frac{0.12}{12}\right]^{-6}$$

$$\Rightarrow d^{(2)} = 2 \left\{ 1 - \left[1 + \frac{0.12}{12}\right]^{-6} \right\}$$

$$\therefore d^{(2)} = 0.1159$$

Assignment

(5) Find the present value of \$25000 to be paid at the end of 26 months at a discount of 4% convertible quarterly. Interpret your result.

Soln:

we know that

$$1 - d = \left[1 - \frac{d(m)}{m} \right]^m$$

$$\Rightarrow d = 1 - \left[1 - \frac{d(m)}{m} \right]^m$$

$$= 1 - \left[1 - \frac{0.04}{4} \right]^4$$

$$= 1 - (0.99)^4$$

$$= 0.0394$$

For 24 months (2 years)

$$A(2) = P(1 - d)^{-t}$$

$$\Rightarrow 25000 = P(0.96)^{-2}$$

$$\Rightarrow P = 25000(0.96)^2$$

$$\therefore P = 23040$$

transmission

for two months,

$$23040 \left(1 - 0.0394 \times \frac{60}{360} \right)$$

Assignment

(5) Find the present value of \$25000 to be paid at the end of 26 months at a discount of 4% convertible quarterly. Interpret your result.

Soln:

$$\text{Quarterly discount rate} = \frac{0.04}{4} = 0.01$$

Total quarters in the 26 months

$$\text{period} = \frac{26}{3} = 8.67 = 8 \frac{2}{3}$$

So, there are 8 complete quarters and 2 remaining months.

$$\begin{aligned} \text{Quarterly discount rate, } d &= 0.01 \\ \text{compound discount factor} &= 1 - d \\ &= 1 - 0.01 \\ &= 0.99 \end{aligned}$$

$$\therefore PV = FV \times (1 - d)^n$$

$$FV = \$25000, \quad (1 - d) = 0.99$$

$$\text{and } n = 8 \frac{2}{3}$$

compound discount for 3 months
is 0.01 on 1%.

discount for 1 month

$$= \frac{0.01 \times 2}{3} = 0.00667$$

for two remaining months

$$(1-d) = \left(1 - \frac{0.02}{3}\right)$$

$$= 0.9933$$

$$\therefore PV = 25000 \times (0.99)^8 \times 0.9933$$
$$= 22914.06$$

Simple discount for remaining
months.

$$PV = FV \times (1 - d \times t)$$

Force of Interest!

The measure of the intensity of interest at a particular moment of time.

By definition,

$$\begin{aligned}\delta t &= \frac{a'(t)}{a(t)} \\ &= \frac{d}{dt} \ln a(t)\end{aligned}$$

$$\begin{aligned}\int_0^t \delta n dt &= \int_0^t \frac{d}{dt} \ln a(t) dt \\ &= \ln [a(t)] - \ln a(0)\end{aligned}$$

$$\begin{aligned}a(t) &= e^{\int_0^t \delta n dt} \\ &= e^{\ln [a(t)] - \ln a(0)}\end{aligned}$$

$$a(t) = (1+i)^t$$

$$\begin{aligned}\delta t &= \frac{a'(t)}{a(t)} = \frac{(1+i)^t \ln(1+i)}{(1+i)^t} \\ &= \ln(1+i)\end{aligned}$$

$$1+i = e^{\delta}$$

$$(1+i)^t = e^{\delta t}$$

$$a(t) = (1+it)$$

$$s_t = \frac{a'(t)}{a(t)} = \frac{1}{1+it}$$

$$a(t) = (1-dt)^{-1}$$

$$s_t = \frac{a'(t)}{a(t)} = \frac{- (1-dt)^{-2} (-d)}{(1-dt)} = \frac{d}{1-dt}$$

Examples:

Find the accumulated \$200 invested for 5 years if force of interest is $s_t = \frac{1}{8+t}$

$$s_t = \frac{1}{8+t} \quad \int_0^5 \frac{1}{8+t} dt$$

$$A(5) = A(0) \cdot e^{\int_0^5 \ln(8+t) dt}$$

$$= 200 \times e$$

$$= 200 \times [(\ln 13) - \ln 8]$$

$$= 200 \times \exp[\ln 13 - \ln 8]$$

$$= 200 \times \exp\left[\ln \frac{13}{8}\right]$$

$$= 200 \times \frac{13}{8}$$

$$= 325$$

Assignment

(8) A deposit of \$25 is invested at time 2 years, using a force of interest of $\delta_t = 0.02t$, find the accumulated value of this payment at the end of 5 years.

Soln:

$$\begin{aligned} A(2,5) &= A(0) \exp \left[\int_2^5 \delta_t dt \right] \\ &= 25 \exp \left[\int_2^5 0.02t dt \right] \\ &= 25 \exp \left[\left(0.02 \cdot \frac{t^2}{2} \right)_2^5 \right] \\ &= 25 \exp \left[(0.01t^2)_2^5 \right] \\ &= 25 e^{0.25 - 0.04} \\ &= \$30.84 \end{aligned}$$

(9) The force of interest, δ_t is,

$$\delta_t = \begin{cases} 0.01 & t \leq 5 \\ 0.02t & 6 \leq t \leq 7 \\ 0.015 & t \geq 7.8 \end{cases}$$

calculate the present value of \$500 payable at time 10.

Soln:

$$PV = A(t) e^{-\int_0^t 2sn dt}$$

$$= A(t) \exp \left[- \left\{ \int_0^{10} 8n dt \right\} \right]$$

$$= 500 \times \exp \left[- \left\{ \int_0^5 8n dt + \int_5^7 8n dt + \int_7^{10} 8n dt \right\} \right]$$

$$= 500 \times \exp \left[- \left\{ \int_0^5 0.1 dt + \int_5^7 0.02 t dt + \int_7^{10} 0.05 dt \right\} \right]$$

$$= 500 \times \exp \left[- \left\{ (0.1)t \Big|_0^5 + (0.02 \frac{t^2}{2}) \Big|_5^7 + (0.05t) \Big|_7^{10} \right\} \right]$$

$$= 500 \times \exp \left[- \left\{ 0.5 + (0.49 - 0.36) + (0.5 - 0.4) \right\} \right]$$

$$= 500 \times \exp \left[- 0.79 \right]$$

$$= \cancel{\$211.58} \quad \$240.95$$

(10) The force of interest at time t , where $0 \leq t \leq 10$ is given by $\delta_t = 0.01$, calculate the accumulated value at time 10 of the payment stream $10e^{0.05t}$, which is received between time 5 and 10.

Soln:

$$\begin{aligned}
 AV_{t=10} &= \int_5^{10} 10e^{0.05t} \exp\left(\int_t^{10} 0.01 ds\right) dt \\
 &= \int_5^{10} 10e^{0.05t} \exp\left([0.01s]_t^{10}\right) dt \\
 &= \int_5^{10} 10e^{0.05t} \exp(0.1 - 0.01t) dt \\
 &= \int_5^{10} 10e^{0.05t} \cdot e^{0.1 - 0.01t} dt \\
 &= 10e^{0.1} \int_5^{10} e^{0.04t} dt \\
 &= 10e^{0.1} \left[\frac{e^{0.04t}}{0.04} \right]_5^{10} \\
 &= 10e^{0.1} \left[\frac{e^{0.4} - e^{0.2}}{0.04} \right] \\
 &= 74.72
 \end{aligned}$$

$$AV_{t=b} = \int_a^b s(t) \exp \left[\int_t^b s(s) ds \right] dt$$

$$PV_{t=a} = \int_a^b s(t) \exp \left[- \int_a^t s(s) ds \right] dt$$

Example!

The force of interest is constant

$$s(t) = 0.01t + 0.04 \quad ; \quad 0 \leq t \leq 5$$

calculate the present value at time 0 of the payment stream $0.5t + 2$, which is received between time 0 and 5.

Soln:

$$PV_{t=0} = \int_0^5 (0.5t + 2) \exp \left[- \int_0^t (0.01s + 0.04) ds \right] dt$$

$$= \int_0^5 (0.5t + 2) \exp \left[- \left\{ 0.01 \frac{s^2}{2} + 0.04s \right\}_0^t \right] dt$$

$$= \int_0^5 (0.5t + 2) \exp \left[- (0.005t^2 + 0.04t) \right] dt$$

(2)

A loan of 3000 is taken out on June 23, 1997. If the force of interest is 14%, find each of the following:

(a) The value of the loan on June 23, 2002.

(b) The value of $i^{(12)}$.

(c) The value of $i^{(12)}$.

Soln:

(a)

$$\begin{aligned} A(5) &= 3000(1+i)^5 \\ &= 3000(e^{0.14})^5 \\ &= 3000 \times e^{0.7} \end{aligned}$$

$$= 6041.26$$

(b)

$$\begin{aligned} 1+i &= e^8 \\ \Rightarrow i &= e^8 - 1 = 0.15027 \\ &= e^{0.14} - 1 \end{aligned}$$

(c)

$$\begin{aligned} \left(1 + \frac{i^{(12)}}{12}\right)^{12} &= 1+i \\ &= e^{0.14} \\ \Rightarrow i^{(12)} &= 12(e^{0.14/12} - 1) \\ &= 0.19082 \end{aligned}$$

Annuities

An annuity may be defined as a series of payments made at equal intervals of time.

- a car loan being repaid with equal monthly installments
- a life insurance policy being purchased with monthly premiums

Types of Annuities:

Annuity Immediate: When the payments are made at the end of the payment intervals, then the annuity is called ordinal annuity or immediate annuity.

Annuity Due: When the payments are made at the beginning of the payment intervals, then the annuity is called annuity due.

(*) show that,

$$a_{\overline{n}|} = \frac{1 - v^n}{i} \quad \text{and} \quad s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

Proof:

Let the present value of a payment of 1 at the end of the first year is v . The present value of a payment of 1 at the end of the 2nd year is v^2 and similarly the present value of a payment of 1 at the end of the n th year is v^n .

$\therefore$ Total present value,

$$\begin{aligned} a_{\overline{n}|} &= v + v^2 + \dots + v^n \\ &= v [1 + v + v^2 + \dots + v^{n-1}] \\ &= v \cdot \frac{(1 - v^n)}{1 - v} \\ &= v \cdot \frac{(1 - v^n)}{d} \\ &= v \cdot \frac{(1 - v^n)}{iv} \\ &= \frac{1 - v^n}{i} \end{aligned}$$

$$v = \frac{1}{1+i} = (1+i)^{-1} \text{ and } (1+i)^{-m} = \frac{1}{(1+i)^m}$$

Again, the accumulated value of a payment of 1 at the end of the first year is $(1+i)^{n-1}$. Similarly, the process is continued until the accumulated value of a payment of 1 at the end of the n th year is just 1.

$\therefore$ Total accumulated value

$$S_n = (1+i)^{n-1} + (1+i)^{n-2}$$

$$+ \dots + (1+i) + 1$$

$$= \frac{(1+i)^n - 1}{(1+i) - 1}$$

$$= \frac{(1+i)^n - 1}{i}$$

(shown)

Example!

(1) John borrows 1500 from a finance company and wishes to pay it back with equal annual payments at the end of the next 10 years. If $i = 0.17$, what should be his annual payment?

Soln:

Let x be the annual payment.
Since the present value of 10 payments of 1 is $a_{\overline{10}|}$, the present value of 10 payments of x will be $x \cdot a_{\overline{10}|}$.

$$\therefore x \cdot a_{\overline{10}|} = 1500 \quad \dots (i)$$

$$a_{\overline{10}|} = \frac{1 - v^{10}}{i} \quad \dots (ii)$$

$$\text{and } v = \frac{1}{1+i} \quad \dots (iii)$$

$$\text{where } i = 0.17$$

$$\therefore v = \frac{1}{1+0.17} = 0.8547$$

$$\therefore x = \frac{1500}{a_{\overline{10}|}} = \frac{1500 \cdot i}{1 - (1+i)^{-10}} = 191.17$$

$$\therefore a_{\overline{10}|} = \frac{1 - (0.855)^{10}}{0.17}$$

$$= 4.654$$

$$\therefore X = \frac{1500}{a_{\overline{10}|}} = \frac{1500}{4.654} = 322.30$$

(2) Eileen deposits 2000 in a bank account every year for 11 years, if $i = 0.06$, how much has he accumulated at the time of the last deposit?

Soln:

Let X be the last deposit.

Since each deposit is 2000, the accumulated value will be

$$X = (2000, s_{\overline{11}|})$$

$$s_{\overline{n}|} = \frac{(1+i)^n - 1}{i}$$

$$\Rightarrow s_{\overline{11}|} = \frac{(1+0.06)^{11} - 1}{0.06}$$

$$= 14.97$$

$$\therefore X = 2000 \times 14.97 = 29940$$

loan to pay
amount PV

(3) Elroy takes out a 5000 loan to buy a car. No payments are due for the first 8 months, but beginning with the 9th month, he must make 60 equal monthly payments. If $i = 0.18$ find,

(i) The amount of each payment

(ii) The amount of each payment if there is no payment free period.

Soln:

(i) Let a monthly rate of interest j ,

$$(1+j)^m = 1+i$$

$$\Rightarrow j = (1+i)^{1/m} - 1$$

$$\Rightarrow j = (1+0.18)^{1/12} - 1$$

$$\therefore j = 0.01389$$

Now,

$$X \cdot a_{\overline{60}|} = 5000(1+j)^8$$

$$\Rightarrow X \cdot a_{\overline{60}|} = 5583.37$$

$$\Rightarrow X = \frac{5583.37}{a_{\overline{60}|}}$$

$$a_{\overline{60}|} = \frac{1-v^m}{j}$$

$$v = \frac{1}{1+j} = \frac{1}{1+0.01389}$$

$$= 0.9863$$

$$\therefore a_{\overline{60}|} = \frac{1 - (0.9863)^{60}}{0.01389}$$

$$= 40.53$$

$$\therefore X = \frac{5583.37}{40.53}$$

$$= 137.76$$

If there is no payment
free period then,

$$X \cdot a_{\overline{60}|} = 5000$$

$$\therefore X = 123.37$$

Derivation of $\ddot{a}_{\overline{n}|i}$

Let the present value of the payment 1 at the beginning of the first year is v_0 . The present value of the payment 1 at beginning of the second year is v_1 .

This procedure is continued until the present value of a payment of 1 at the beginning of the n th year is v_{n-1} .

Total present value,

$$\ddot{a}_{\overline{n}|i} = 1 + v + v^2 + \dots + v^{n-1}$$

$$= \frac{1 - v^n}{1 - v}$$

$$= \frac{1 - v^n}{d}$$

where d is the effective rate of discount.

$$\frac{1}{d} = \frac{1}{1 - v}$$

The value of this annuity at time 0 is 1. The value of the annuity at time 1 is v .

Total accumulated value

$$\begin{aligned} S_m &= C(1+i)^n + C(1+i)^{n-1} + \dots + C(1+i)^2 + C(1+i) + C \\ &= C(1+i) [1 + (1+i) + (1+i)^2 + \dots + (1+i)^{n-1}] \end{aligned}$$

$$= C(1+i) \cdot \frac{(1+i)^n - 1}{i}$$

$$= \frac{C(1+i)^n - 1}{\frac{i}{1+i}}$$

$$= \frac{C(1+i)^n - 1}{i \cdot v}$$

$$= \frac{C(1+i)^n - 1}{d}$$

Perpetuity: An annuity whose payments continue forever

$$a_{\infty} = \frac{1}{i}$$

$$\ddot{a}_{\infty} = \frac{1}{d}$$

The value of this annuity one year before the first payment is a_{∞}

A deferred annuity is one for which the first payment starts some time in the future.

$\ddot{a}_{\overline{n}|}$ represents the value of perpetuity at the time of the first payment

Show that

$$(a) \ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$$

$$(b) \dot{s}_{\overline{n}|} = s_{\overline{n-1}|} + 1$$

proof:

$$R.H.S = 1 + a_{\overline{n-1}|}$$

$$= 1 + \frac{1 - v^{n-1}}{i}$$

$$= 1 + \frac{1 - v^n \cdot \frac{1}{v}}{i}$$

$$= \frac{1 + i - v^n \cdot \frac{1}{v}}{i}$$

$$= \frac{(1+i) - v^n \cdot (1+i)}{i}$$

$$= \frac{(1+i)(1 - v^n)}{i}$$

$$= \frac{1 - v^n}{\frac{i}{1+i}} = \frac{1 - v^n}{d}$$

$$= \ddot{a}_{\overline{n}|} \quad (\text{proved})$$

(b)

$$R.H.S = 3\overline{m-1} - 1$$

$$= \frac{(1+i)^{m+1} - 1}{i} - 1$$

$$= \frac{(1+i)^m (1+i) - 1 - i}{i}$$

$$= \frac{(1+i)^m (1+i) - (1+i)}{i}$$

$$= \frac{(1+i) [(1+i)^m - 1]}{i}$$

$$= \frac{(1+i)^m - 1}{\frac{i}{1+i}}$$

$$= \frac{(1+i)^m - 1}{i}$$

$$= 3\overline{m}$$

(*)

$$(a) \ddot{a}_m = a_m + 1 - v^n$$

$$R.H.S = a_m + 1 - v^n$$

$$= \frac{1 - v^n}{i} + 1 - v^n$$

$$= \frac{(1 - v^n) + i(1 - v^n)}{i} =$$

$$= \frac{(1 - v^n)(1 + i)}{i}$$

$$= \frac{(1 - v^n)}{i / (1 + i)}$$

$$= \frac{1 - v^n}{d}$$

$$= \ddot{a}_m$$

$$= L.H.S$$

$$(b) \dot{s}_m = s_m - 1 + (1 + i)^m$$

$$R.H.S = s_m - 1 + (1 + i)^m$$

$$= \frac{(1 + i)^m - 1}{i} + \{ (1 + i)^m - 1 \}$$

$$= \frac{[(1 + i)^m - 1](1 + i)}{i}$$

$$= \frac{(1 + i)^m - 1}{i / (1 + i)}$$

$$= \frac{(1 + i)^m - 1}{d} = \dot{s}_m$$

$$= L.H.S$$

(*)

$$(a) \quad a_{m+n} = a_m + v^m a_m = 1 \text{ (L.H.S.)}$$

$$\text{R.H.S.} = a_m + v^m a_m$$

$$= \frac{1-v^m}{i} + v^m a_m$$

$$= \frac{1-v^m}{i} + v^m \frac{1-v^m}{i}$$

$$= \frac{1-v^m}{i} + \frac{v^m - v^{m+1}}{i}$$

$$= \frac{1-v^m + v^m - v^{m+1}}{i}$$

$$= \frac{1-v^{m+1}}{i}$$

$$= \text{L.H.S.}$$

$$(b) \quad a_{m-n} = a_m - v^m s_m$$

$$\text{R.H.S.} = a_m - v^m s_m$$

$$= \frac{1-v^m}{i} - v^m \frac{(1+i)^m - 1}{i}$$

$$= \frac{1 - \left(\frac{1}{1+i}\right)^m}{i} - \left(\frac{1}{1+i}\right)^m \frac{(1+i)^m - 1}{i}$$

$$= \frac{(1+i)^m - 1}{i(1+i)^m}$$

$$= \frac{(1+i)^{m-1}}{i(1+i)^m} - \frac{(1+i)^{n-1}}{i(1+i)^n}$$

$$= \frac{1}{(1+i)^m} \left[\frac{(1+i)^{m-1}}{i} - \frac{(1+i)^{n-1}}{i} \right]$$

$$= \frac{1}{i(1+i)^m} \left[(1+i)^{m-1} - (1+i)^{n-1} \right]$$

$$= \frac{1}{i} \left[1 - \frac{(1+i)^{n-m}}{(1+i)^m} \right]$$

$$= \frac{1}{i} \left[1 - (1+i)^{n-m} \right]$$

$$= \frac{1}{i} \left[1 - \left(\frac{1}{1+i} \right)^{m-n} \right]$$

$$= \frac{1}{i} \left[1 - v^{m-n} \right]$$

$$= \frac{1 - v^{m-n}}{i}$$

$$= a_{\overline{m-n}|i}$$

$$= L.H.S$$

$$ce) \overline{s_{m+n}} = \overline{s_m} + (1+i)^m \overline{s_n}$$

$$\Rightarrow R \cdot t \cdot S = \overline{s_m} + (1+i)^m \overline{s_n}$$

$$= \frac{(1+i)^m - 1}{i} + (1+i)^m \cdot \frac{(1+i)^n - 1}{i}$$

$$= \frac{(1+i)^m - 1}{i} + (1+i)^{m+n} - (1+i)^m$$

Force of interest in real life

- Bond pricing
- Loan amortization
- Pension fund management
- Insurance pricing
- Investment appraisal
- Economic modeling
- Personal finance planning
- Real estate valuation

$$1+i = \left[1 + \frac{i^m}{m} \right]^m$$

$$\Rightarrow i^2$$

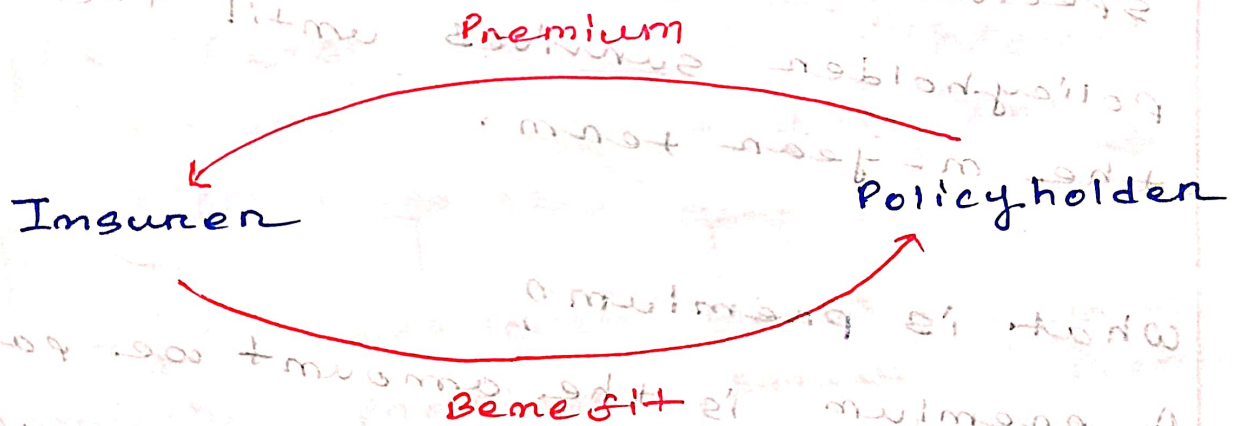
(4)

Life Insurance

What is life insurance?

A life insurance is a contract between the insurer and another party known as the policyholder.

In return for a payment of premiums, the insurer will pay a predetermined amount of money, known as a death benefit, upon the death of the policyholder.



Types of life insurance:

① Whole life insurance:

A policy where a fixed amount, known as the face value (death benefit), is paid to the beneficiary at the end of the year of death,

regardless of when death occurs!

② Term insurance

A policy where the face value is paid only if death occurs within a prescribed period.

③ An n -year endowment insurance!

A policy where the face value is paid if death occurs within a specified n -year period, or if the policyholder survives until the end of the n -year term.

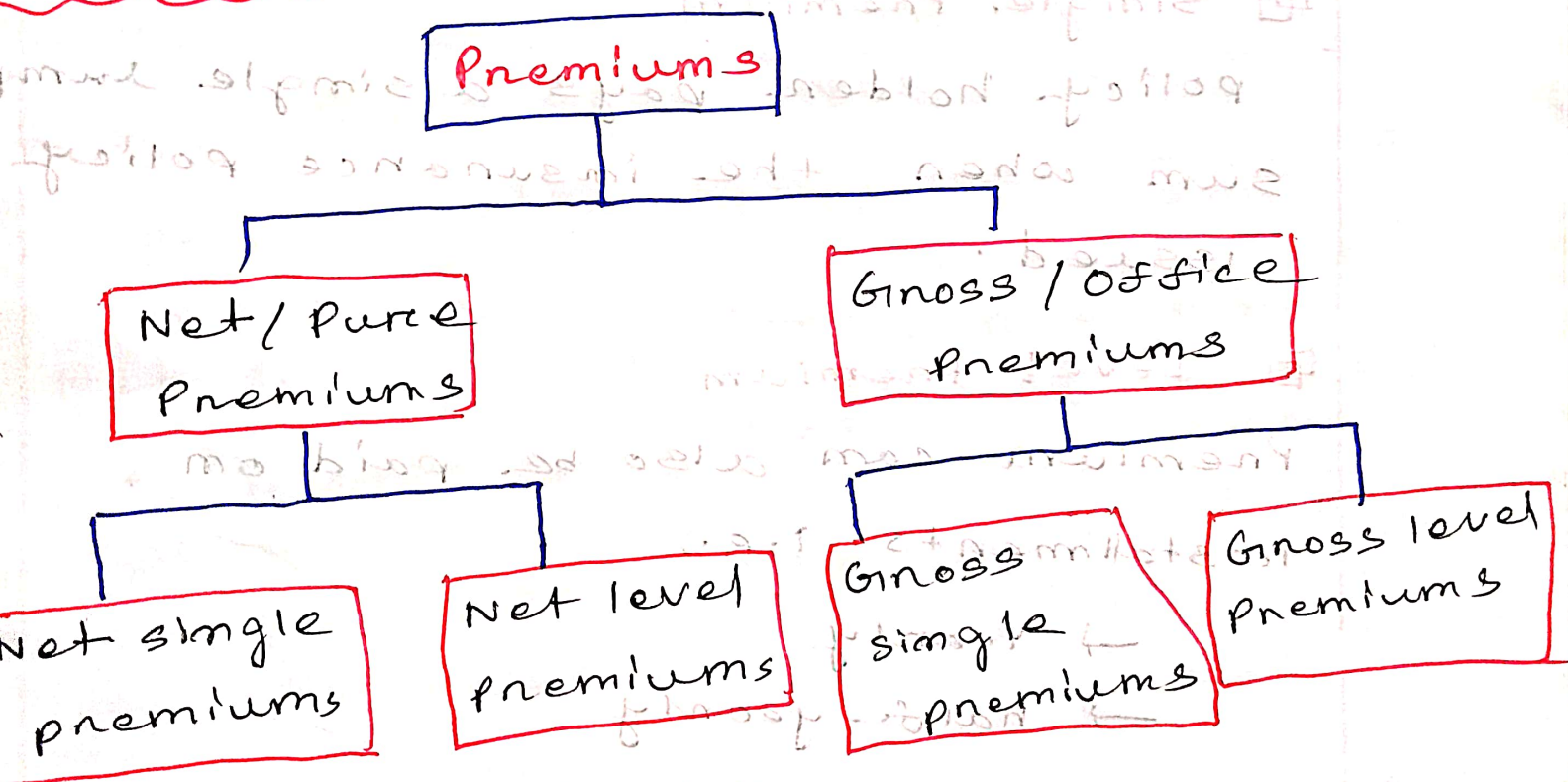
What is premium?

A premium is the amount we pay to an insurance company in exchange for coverage under a specific policy.

This payment keeps our policy active and ensures that the insurer will provide financial protection as outlined in our agreement.

Premiums can be paid on various schedules, such as monthly, quarterly, semiannually, or annually, depending on the terms of the policy and own preferences.

Types of Premiums:



• Net Premium: Net premium is based

- on:
- Mortality and
 - Interest rate

• **Gross Premium:** Gross premium is based on:

- Mortality
- Interest rates
- Expenses and
- Bonus loading

▣ **Single Premium**

policy holder pays a single lump sum when the insurance policy is issued.

▣ **Level Premium**

Premium can also be paid on installments i.e.,

- Yearly
- half-yearly
- quarterly or
- monthly

Calculation of Net single premium:

• Whole life insurance

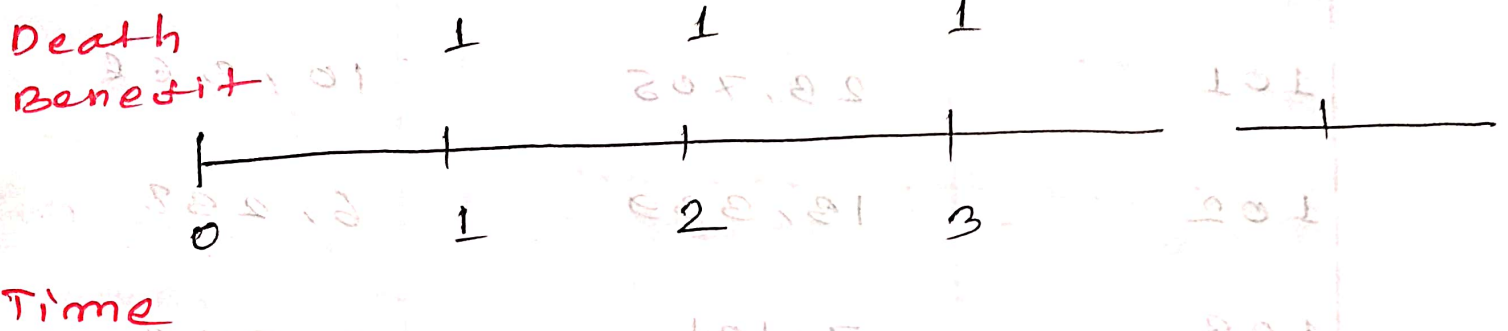


Figure-1 Time diagram

of discrete whole life insurance

Net single premium for whole life insurance (with face value 1)

$$A_x = \sum_{t=0}^{\omega-1-x} {}_t p_x v^{t+1}$$

$$= 1 \times {}_0 p_x v + 1 \times {}_1 p_x v^2 + \dots + 1 \times {}_{\omega-1-x} p_x v^{\omega-x}$$

where, ω = limiting age (110)

${}_k p_x$ = A person aged x years will survive k years

${}_1 q_x$ = A person aged x years will die next year

Example

x	$\ln x$	$d \ln x$
100	40,049	16,344
101	23,705	10,368
102	13,339	6,238
103	7,101	3,543
104	3,558	1,890
105	1,668	941
106	727	435
107	292	184
108	108	72
109	36	25
110	11	11
111	0	0

Using the provided life table, calculate the net single premium for a whole life insurance policy issued to an individual aged 100 and $i = 0.06$.

Ams:

Hence, face value = 1, $i = 0.06$
and $x = 100$

Discount factor,

$$v = \frac{1}{1+i} = \frac{1}{1+0.06} = 0.943$$

$$w = \frac{111}{10}$$

$$\therefore A_{100} = \sum_{t=0}^{10} {}_t p_x q_{x+t} v^{t+1}$$

$$= \sum_{t=0}^{10} \frac{{}_d n_{x+t} v^{t+1}}{{}_x n} = \frac{A'_{100}}{{}_x n}$$

$$\begin{aligned} A'_{100} &= 16344 \times (0.943)^1 + 10366 \times (0.943)^2 \\ &+ 6238 \times (0.943)^3 + 3543 \times (0.943)^4 \\ &+ 1800 \times (0.943)^5 + 941 \times (0.943)^6 \\ &+ 435 \times (0.943)^7 + 184 \times (0.943)^8 \end{aligned}$$

$$+ 72 \times (0.94379) + 25 \times (0.94379)^{10} \\ + 11 \times (0.94379)^{11}$$

$$= 35199.65637$$

PV of Total claim

$$\text{Net single premium} = \frac{\text{Total net of insured person}}{\text{Total net of insured person}}$$

$$= \frac{A'_{100}}{2100} \\ = \frac{35199.65637}{40049}$$

$$= 0.8789147$$

$$\text{If face value} = 10000$$

$$\text{NSP} = \frac{10,000 A'_{100}}{2100}$$

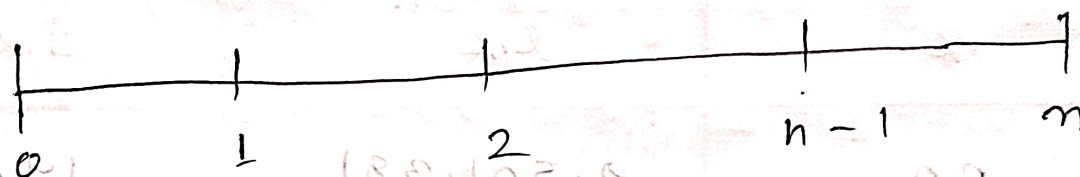
$$= \frac{10,000 \times 35199.65637}{40,049}$$

$$= 8789.15$$

cc)

• Term Insurance

Death
Benefit



Time

Figure 1.2 cash flow of Term insurance

Net single premium for term insurance

$$A_{\overline{x:n}|} = \sum_{t=0}^{n-1} p_x q_{x+t} v^{t+1}$$

$$= \sum_{t=0}^{n-1} d_{x+t} v^{t+1}$$

$$\sum_{t=0}^{n-1} v^{t+1} = \frac{1}{1-v} A_{\overline{x:n}|}$$

$$v \times 1.06 + v \times 1.06 + \dots + v \times 1.06 = v \times 1.06 \times n$$

Example:

calculate net single premium of a 3 year term insurance on (30) of 1000. Given, $i = 0.06$.

x	l_x	dx
30	9,501,381	14,527
31	9,486,854	15,263
32	9,471,591	16,069
33	9,455,522	

Solution:

Face value = 1000

interest rate, $i = 0.06$, $v = \frac{1}{1+i}$

$$n = 3$$

$$x = 30$$

$$\begin{aligned} \therefore A_{\overline{30:3}|} &= \sum_{t=0}^2 \frac{dx_{t+1}}{l_{30}} v^{t+1} \\ &= \frac{dx_{31} v^1 + dx_{32} v^2 + dx_{33} v^3}{l_{30}} \end{aligned}$$

$$= \frac{1}{9,501,381} \left[14,527 \times 0.943 + 15,263 \times (0.943)^2 + 16,049 \times (0.943)^3 \right]$$

$$= 0.0004288$$

∴ Net single premium =

$$= 1000 \times A_{30:3}$$

$$= 1000 \times 0.004288$$

$$= 4.288$$

• n year endowment insurance policy

Death
Benefit
Survival
Benefit

Time

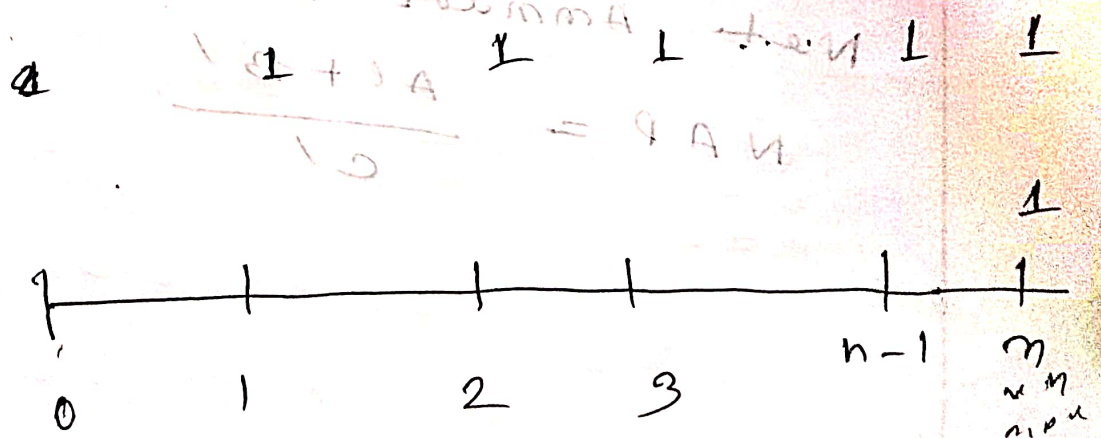


Figure 3 An n year endowment insurance policy

PV of death claim, $A' = \text{Face value}$

$$A' \times \sum_{t=0}^{n-1} d_{x+t} v^{t+1}$$

PV of living person claim,

$$B' = \text{Face value} \times (d_{x+n-1} - d_{x+n-1})$$

$$A' \times v^n$$

Net single premium,

$$NSP = \frac{A' + B'}{d_x}; \quad x = \text{starting age}$$

$$C' = \sum_{k=0}^{n-1} v^k l_{x+k}$$

Net Annual Premium,

$$NAP = \frac{A' + B'}{C'}$$

Example:

Calculate the (1) Net single premium and (2) Net annual premium for five years endowment policy of Tk. 5,000 from the following information

x	lx	dx
35	30,550	4,196
36	26,354	3,285
37	23,069	2,562
38	20,507	1,978
39	18,529	1,164

$$i = 0.02$$

Solution:

calculation of present value of death claim:

$i = 0.02$ face value = 5000
 $\therefore v = \frac{1}{1+i} = \frac{1}{1+0.02} = 0.9801$

$$5000 A_{\overline{5}|0.02} = \sum_{t=0}^{n-1} dx + t v^{t+1} \times 5000$$

$\Rightarrow$ Let,

$$A' = \sum_{t=0}^{n-1} dx + t v^{t+1} \times 5000$$

(p.v of death claim)

$$= \sum_{t=0}^4 dx + t v^{t+1} \times 5000$$
$$= 5000 [4196 \times (0.9801) + 3285 \times (0.9801)^2 + 2562 \times (0.9801)^3 + 1978 \times (0.9801)^4 + 1164 \times (0.9801)^5]$$

$$= 62774679.66$$

calculation of PV of living person claim:

No. of living person at the end of 5 year = $(18529 - 11642)$

$$= 17365$$

Total living person claim =

$$= 5000 \times 17365$$

$$= 86825000$$

Present value of living person claim, B' =

$$86825000 \times 0.98075$$

$$= 78482923.18$$

Net single premium:

$$x = 35, \quad 235 = 30550$$

$$NSP = \frac{A' + B'}{235}$$

$$= 4623.82$$

calculation of present value of Tk. 1 receivable annually from each living policy holder,

$$P1 = \sum_{k=0}^{n-1} v^k \text{ Tk. } 1 = \sum_{k=0}^4 v^k \text{ Tk. } 1$$

$$= 135 v^0 + 136 v^1 + 137 v^2 + 138 v^3 + 139 v^4$$

$$= 114023.9716$$

Net Annual Premium,

$$\text{NAP} = \frac{A1 + B1}{C1}$$

$$= 1229.14 \text{ Tk.}$$

(3)

Amortization and Sinking Fund

- Amortization is the process of repaying a loan by equal periodic payments.
- This is an example of annuity.
- Our goal is to find the outstanding principal.
- Two methods to compute P_t :
 - prospective
 - retrospective
- **Prospective Method:** outstanding principal at any point in time is equal to the present value at that date of all remaining payments.
- **Retrospective Method:** outstanding principal is equal to the original principal accumulated to that point in time minus the accumulated value of all payments previously made.

Amortization Schedule:

It is a detailed table that breaks down each periodic payment of a loan into its two components:

→ Interest

→ Principal

Example:

Formula:

$$A = P \cdot \frac{i(1+i)^m}{(1+i)^m - 1}$$

A = periodic payment amount

P = amount of principal

i = periodic interest rate

m = total number of payments

① Consider a 5000 loan at 12% per year to be repaid by 5 annual payments. Construct an amortization schedule.

Solution!

Given,

Interest rate, $i = 12\% = 0.12$

Total number of payments,

$$n = 5$$

Amount of principal, $P = 5000$

Amount of each payment,

$$A = P \cdot \frac{i(1+i)^n}{(1+i)^n - 1}$$

$$= 5000 \times \frac{0.12(1+0.12)^5}{(1+0.12)^5 - 1}$$

$$= 1387.05$$

Amortization Schedule for this loan,

(1)	(2)	(3)	(2) - (3) = (4)	(5)
Duration	Payment	Interest	Principal Repaid	Outstanding Principal
0				5000
1	1387.05	$5000 \times 0.12 = 600$	$1387.05 - 600 = 787.05$	$5000 - 787.05 = 4212.95$
2	1387.05	505.55	881.50	3331.45
3	1387.05	399.77	987.28	2344.17
4	1387.05	281.30	1105.75	1238.42
5	1387.05	148.61	1238.44	0

(2) A 1000 loan is repaid by annual payments of 150, plus a smaller final payment. If $i = 0.11$ and the first payment is made one year after the time of the loan, construct an amortization.

Duration	Payment	Interest	Principal Repaid	Outstanding Principal
0				1000
1	150	110	40	960
2	150	105.6	44.4	915.6
3	150	100.72	49.28	866.32
4	150	95.29	54.71	811.61
5	150	89.28	60.72	750.89
6	150	82.60	67.40	683.49
7	150	75.18	74.82	608.67
8	150	66.95	83.05	525.62
9	150	57.82	92.18	433.44
10	150	47.68	102.32	331.12
11	150	36.42	113.58	217.54
12	150	23.93	126.07	91.47
13	100.53	10.06	91.47	0

(3) A loan of \$15000 at 4.5% compounded monthly with monthly payments for 3 years.

Solution:

The effective monthly interest rate is, $i = 0.045/12$

$$= 0.00375$$

The number of payment is,

$$n = 12 \times 3 = 36$$

the principal amount is,

$$P = 15000$$

$\therefore$ Amount of each monthly payment,

$$A = P \cdot \frac{i(1+i)^n}{(1+i)^n - 1}$$

$$= 446.201$$

Amortization schedule for first 3 months:

Duration	payment	Interest	Principal repaid	Outstanding principal
0				15000
1	446.20	56.25	389.95	14610.05
2	446.20	54.79	391.41	14218.64
3	446.20	53.32	392.88	13825.76
⋮	⋮	⋮	⋮	⋮
36	446.20	---	---	0

Sinking Fund:

- A sinking fund is a reserve account where funds are regularly deposited to repay a bond or debt at maturity.
- It is often managed by a trustee and ensures the issuer can meet repayment obligations.

Amount of sinking fund payment,

$$X = P \frac{i}{(1+i)^n - 1}$$

Example:

(1) Julie wants to buy a \$24,000 car in 6 years. How much money she deposits at the end of each quarter in an account paying 5% compounded quarterly so that she will have enough to pay for the car?

Solution:

Hence,

$$P = 24000 \quad (\text{Future value})$$

Effective quarterly interest,

$$i = 0.05 / 4 = 0.0125$$

$$n = 6 \times 4 = 24 \quad [\text{Total number of deposits}]$$

Amount of each quarterly
deposit,

$$X = 24000 \times \frac{0.0125}{(1+0.0125)^{24} - 1}$$
$$= \$863.68$$

(2) \$65,000; money earns 6%
compounded ^{semi} annually for 4.5
years.

Solution:

Future values, $P = \$65,000$

$$i = 0.06/2 = 0.03$$

$$n = 2 \times 4.5 = 9$$

Amount of each payment made
into a sinking fund,

$$X = 65000 \times \frac{0.03}{(1+0.03)^9 - 1}$$

$$= \$6398.20$$

Sinking Fund Schedule

Payment Index	Payment Amount	Interest paid	Deposit in sinking fund	Sinking fund balance
0				
1	6398.20	100		
2	6398.20	100		
3	6398.20	100		
4	6398.20	100		
5	6398.20	100		
6	6398.20			
7	6398.20			
8	6398.20			
9	6398.20			
Total				

(3) John borrows 15,000 from a trust company at 17% effective annually. He agrees to pay the interest annually, and to build up a sinking fund which will repay the loan at the end of 15 years. If the sinking fund accumulates at 12% annually, find,

(a) The annual interest payment is $iL = 0.17 \times 15000 = 2550$

(b) The annual sinking fund payment is,

$$X = \frac{15000 \times 0.12}{(1 + 0.12)^{15} - 1} = 402.36$$

(c) His total annual outlay,

$$T = iL + X$$

$$= 2550 + 402.36$$

$$= 2952.36$$

(d)

The amortization payment,

$$A = \frac{15000 \times (1 + 0.17)^{15} \times 0.17}{(1 + 0.17)^{15} - 1}$$
$$= 2817.33$$

(u) Helen wishes to borrow 7000 to buy a car. Our lender offers a loan in which the principal is to be repaid at the end of 5 years. In the meantime interest at 11% effective is to be paid on the loan, and the borrower is to accumulate her principal by means of annual payments into a sinking fund earning 8% effective. Another lender offers a loan, with the first of the annual payments due one year. Find the rate of interest, i , that this second lender can charge in

order that Helen finds the two offers equally attractive.

Solution:

Helen's annual interest payment is,

$$iL = 0.11 \times 7000 = 770, \text{ and}$$

her annual sinking fund payment

X is given by,

$$X = \frac{pi}{(1+i)^n - 1}$$

($i = 0.08$
for sinking
fund)

$$= \frac{7000 \times 0.08}{(1+0.08)^5 - 1}$$

$$= 1193.19$$

Helen's total annual outlay is

$$iL + X = 770 + 1193.19$$

$$= 1963.19$$

We can find i for second offer to make the two offers equally attractive:

$$1963.19 = \frac{7000 \times i \times (1+i)^5}{(1+i)^5 - 1}$$

$$\Rightarrow 1963.10 \text{ a } 57 = 7000$$

$$\Rightarrow \text{a } 57 = 3.5656.25$$

Linear interpolation,

$$\text{for } i = 0.124, \text{ a } 57 = 3.569339$$

$$i = 0.125, \text{ a } 57 = 3.560568$$

$$i = 0.124 + \left[\frac{3.565625 - 3.569339}{3.560568 - 3.569339} \right] [0.125 - 0.124]$$

$$= 0.12442$$

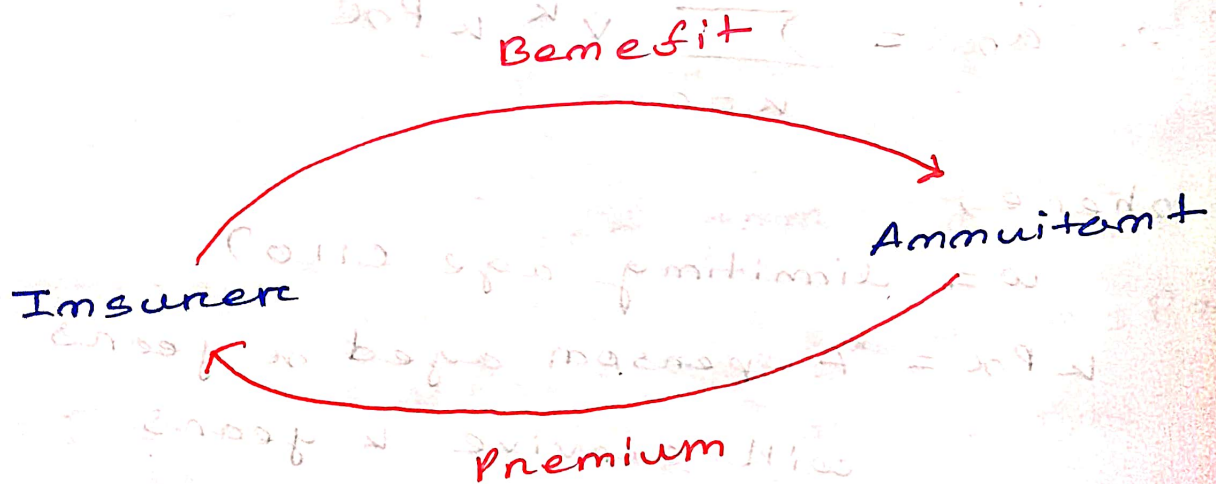
(5)

Life Annuity

What is life annuity?

A life annuity is a contract between an insurance company, known as the insurer, and another party, known as the annuitant.

In return of payment(s) paid by the annuitant, the insurer will provide a sequence of payments, known as annuity benefits of prescribed amounts and at prescribed times.



Types of life annuity:

- whole life annuity

An annuity for which benefits continue for as long as the annuitant lives.

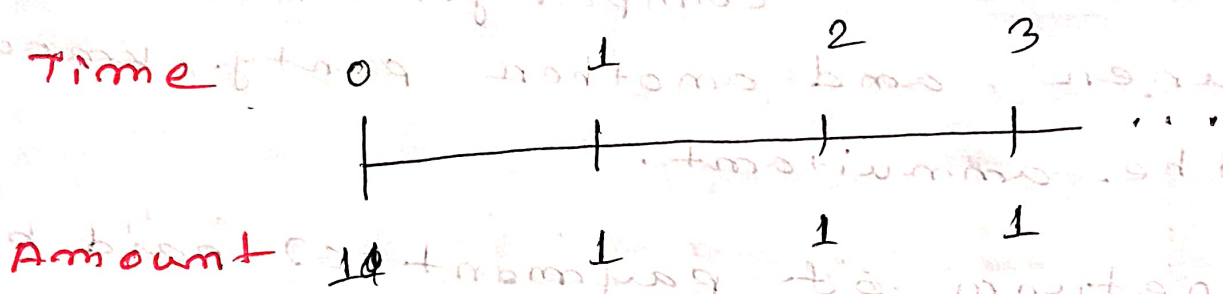


Fig: 1 whole life annuity due

$$\ddot{a}_x = 1 + v p_x + v^2 {}_2p_x + v^3 {}_3p_x + \dots$$

$$\therefore \ddot{a}_x = \sum_{k=0}^{\omega-1-x} v^k {}_k p_x$$

where,

ω = limiting age (110)

${}_k p_x$ = A person aged x years will survive k years

$$m + {}_k p_x = m p_x \cdot {}_k p_{x+m}$$

• **Term (Temporary) life annuity:**

An annuity for which benefit payments are made for a fixed number of years and then cease is known as a temporary life annuity.

Time	0	1	2	3	...	$n-1$	n
					...		
Amount	1	1	1	1	...	1	0
		v	v^2	v^3	...	v^{n-1}	
Discount	1						
		p_x	${}_2p_x$	${}_3p_x$	...	${}_{n-1}p_x$	
probability	1						

Fig 2: Term annuity due

$$\ddot{a}_{x:\overline{n}|} = 1 + v p_x + v^2 {}_2p_x + v^3 {}_3p_x + \dots + v^{n-1} {}_{n-1}p_x$$

$$= \sum_{k=0}^{n-1} v^k {}_k p_x$$

• Deferred annuity

A contract where the initial payment does not commence for several years

Time	0	1	2	...	$u-1$	u	$u+1$	...
Amount	0	0	0	...	0	1	1	...
Discount	1	v	v^2	...	v^{u-1}	v^u	v^{u+1}	...
Probability	1	${}_1p_x$	${}_2p_x$	...	${}_{u-1}p_x$	${}_up_x$	${}_{u+1}p_x$	...

Fig 3: deferred annuity due

$$u| \ddot{a}_x = v^u {}_up_x + v^{u+1} {}_{u+1}p_x + \dots$$

$$= \sum_{k=u}^{\infty} v^k {}_kp_x$$

$u|$ indicates that payments are deferred by a period of u years

Some important relationship:

- i) $\ddot{a}_x = \ddot{a}_{x:\overline{n}|} + n \cdot \ddot{a}_x$
- ii) $a_x = \ddot{a}_x - 1$
- iii) $a_{x:\overline{n}|} = \ddot{a}_{x:\overline{n}|} - 1 + v^n \cdot {}_n p_x$

Example!

(1) Suppose that a person currently age 65 wants to purchase a 3-year term annuity due with annual payments of \$50,000 each. Given,

$$p_{65} = 0.95, \quad p_{66} = 0.91, \quad p_{67} = 0.87$$

$$i = 7\%, \quad v = 0.935$$

Solution!

$$\ddot{a}_{65:\overline{3}|} = \sum_{k=0}^2 v^k \cdot {}_k p_{65}$$

$$= 1 + 1 \cdot p_{65} \cdot v + 2 \cdot p_{65} \cdot v^2$$

$$= 1 + 0.95 \times 0.935 + 0.95 \times 0.91 \times (0.935)^2$$

$$= 2.64$$

$$NSP = 50,000 \times 2.64 = 132,200.87$$

(2)

An annuity on 400 provides for 1 annually for life, beginning at age 65. Nothing is paid before 65. Annual premiums are payable for 25 years beginning at age 40. The premium reduces by one half after 15 years. Find the expression for initial premium, assuming $\omega = 110$.

Solution:

The annuity pays 1 annually from age 65 until death. The present value at age 65 is:

$$\begin{aligned}\ddot{a}_{65} &= \sum_{k=0}^{\omega-65-1} v^k \cdot {}_k p_{65} \\ &= \sum_{k=0}^{\omega-65-1} v^k \cdot \frac{l_{65+k}}{l_{65}}\end{aligned}$$

where

$v = (1+i)^{-1}$ is the annual discount factor

l_x is the number of individuals alive at age x

$\omega = 110$ is the limiting age

To calculate the present value of an annuity at an earlier age x (i.e., 40) discount the annuity value at the deferred starting age $x+n$ (i.e., 65) back n years and account of the probn of surviving from x to $x+n$

$$APV \text{ at age } x = v^n \frac{l_{x+n}}{l_x} \times (APV \text{ at age } x+n)$$

$$\therefore APV \text{ at age } 40 = v^{25} \frac{l_{65}}{l_{40}} \times \sum_{k=0}^{\infty} v^k \frac{l_{65+k}}{l_{65}}$$

$$= v^{25} \frac{l_{65}}{l_{40}} \times \ddot{a}_{65}$$

Premiums are paid annually from age 40 to 65, with the first 15 years at rate p and the remaining 10 years at rate $\frac{p}{2}$.

$$p \cdot \ddot{a}_{40:\overline{15}|} = p \sum_{k=0}^{14} v^k \frac{l_{40+k}}{l_{40}} \dots (1)$$

$$\frac{p}{2} \cdot v^{15} \cdot \frac{l_{55}}{l_{40}} \ddot{a}_{55:\overline{10}|}$$

$$= \frac{p}{2} v^{15} \frac{l_{55}}{l_{40}} \sum_{k=0}^9 v^k \frac{l_{55+k}}{l_{55}}$$

$$\therefore p \ddot{a}_{40:\overline{15}|} + \frac{p}{2} v^{15} \frac{l_{55}}{l_{40}} \ddot{a}_{55:\overline{10}|}$$

$$= v^{25} \frac{l_{65}}{l_{40}} \ddot{a}_{65}$$

$$\Rightarrow p \left[\ddot{a}_{40:\overline{15}|} + \frac{1}{2} v^{15} \frac{l_{55}}{l_{40}} \ddot{a}_{55:\overline{10}|} \right]$$

$$= v^{25} \frac{l_{65}}{l_{40}} \ddot{a}_{65}$$

$$\therefore p = \frac{v^{25} \frac{l_{65}}{l_{40}} \ddot{a}_{65}}{\ddot{a}_{40:\overline{15}|} + \frac{1}{2} v^{15} \frac{l_{55}}{l_{40}} \ddot{a}_{55:\overline{10}|}}$$

Previous for Amortization and sinking fund

What is yield rate or internal rate of return for an investment?

The yield rate is a discount rate that makes sum of the present values of all future cashflows equal to the initial investment.

Mathematically, for an investment with an initial outlay c_0 and future cash inflows c_t at times t , the IRR is the rate r that satisfies the equation:

$$\sum_{t=0}^n \frac{c_t}{(1+r)^t} = 0$$

where,

c_0 = initial investment

c_t = future cash inflows or outflows

r = internal rate of return

t = time period (in years, months etc)

n = total number of periods

For example, if a loan company lends Harry 10,000 which he pays back by the amortization method 15% per year, then the yield earned by the company is 15%.

Amortization vs Sinking Fund

	Amortization	Sinking Fund
Purpose	Gradual repayment of a loan over time.	Set aside money for a future payment on obligation.
Payments	Fixed payments covering both principal and interest.	Periodic contributions to accumulate funds.
Payment structure	Payments reduce loan balance gradually.	Contributions build up a fund for a lump sum payment.
Usage	Used for loans or mortgages.	Used for saving for future expenses.

$$\begin{aligned}
 {}_{m+n}p_x &= \frac{{}_l x + m + n}{{}_l x} \\
 &= \frac{{}_l x + m}{{}_l x} \cdot \frac{{}_l x + m + n}{{}_l x + m} \\
 &= ({}_m p_x) ({}_n p_{x+m})
 \end{aligned}$$

$$\begin{aligned}
 {}_{m|m}q_x &= {}_m p_x \times (1 - {}_n p_{x+m}) \\
 &= {}_m p_x - ({}_m p_x \times {}_n p_{x+m}) \\
 &= {}_m p_x - {}_{m+n}p_x
 \end{aligned}$$

${}_m p_x$ = prob^y that an individual aged x survives m years
 ${}_{m|m}q_x$ = the prob^y that the individual survives m years but does not survive the subsequent m years.

$v^2 \cdot k \cdot p \cdot n$ $p \cdot 55 + 0$
 $0 \cdot p \cdot n$ $1 \cdot 155$

$$dx = dx + dx + 1 + \dots$$

$$p_n = \frac{dx + 1}{dx} \quad \text{or} \quad n = \frac{dx}{p_n}$$

$$dx = (dx - dx + 1)$$

2021

4.
(c) $i = 0.03$, $m = 3$, $n = 55$

Time	0	1	2
Amount	150	200	250
Discount	v^0	v	v^2
Probability	1	p_n	$2 \cdot p_n$

$$NSP = 150 \times 1 \times 1 + 200 \times v \times p_n + 250 \times v^2 \times 2 \cdot p_n$$

$$= 150 + 200 \times (1.03)^{-1} \times 0.90 + 250 \times (1.03)^{-2} \times (0.90 \times 0.92)$$

$$= 519.87$$

(2)

(b) Deferred annuity, a contract where the initial payment does not commence for several years.

Time	x	$x+1$	$x+2$	$x+n$	$x+n+1$	...
Amount	0	0	0	k	k	...
Discount		v	v^2	v^n	v^{n+1}	...
Probability		${}_1p_x$	${}_2p_x$	${}_np_x$	${}_{n+1}p_x$	...

The present value are ${}_np_x$ (ss)

$$k \cdot n | \ddot{a}_x = k \cdot v^n \cdot {}_np_x + k \cdot v^{n+1} \cdot {}_{n+1}p_x + \dots$$

$$= k \sum_{t=n}^{\infty} v^t \cdot {}_tp_x$$

which is the net single premium.

Annual premium = $\frac{n | \ddot{a}_x}{\ddot{a}_x | n}$

$$\begin{aligned}
 & dx + dx+1 + \dots + dx+n-1 \\
 &= (dx - dx+1) + (dx+1 - dx+2) \\
 &\quad + \dots + (dx+n-1 - dx+n) \\
 &= dx - dx+n
 \end{aligned}$$

Give some examples of Gross Premiums:

(1) Life Insurance policy

A person pays an annual gross premium of \$1200 for a life insurance policy. This includes:

- \$900 for the net premiums
- \$300 for administrative costs, agent + commissions, and profit

(2) Car Insurance

The gross premium for a car insurance policy is \$800 per year. This amount consists of:

- \$600 for the net premium (to cover accident claims)
- \$200 for operational costs, broken fees, and taxes.

(3) Health Insurance:

- A health insurance plan charges a gross premium of \$1500 annually.
- \$1200 for the net premium (to pay medical claims)
 - \$300 for loading charges like policy servicing and profit.

Geometric annuity,

$$AV = P \cdot \frac{(1+i)^n - (1+k)^n}{i - k}$$

P = amount of first payment

$$S_m = \frac{n}{2} [2a + (n-1)d]$$

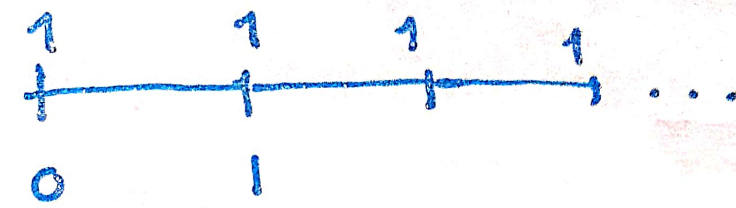
Step 1: Transition to the next state
 cannot be done, as it would

(Simplify the expression)

Next, we have to find the maximum of the function
 for the net profit. The net profit is given by
 the difference between the revenue and the cost.

Let x be the number of units produced. Then the revenue is
 given by $R(x) = 1500x - 0.01x^2$ and the cost is
 given by $C(x) = 1000x + 0.005x^2$.

The profit function is given by $P(x) = R(x) - C(x)$.
 Substituting the expressions for $R(x)$ and $C(x)$, we get
 $P(x) = 1500x - 0.01x^2 - 1000x - 0.005x^2$
 $P(x) = 500x - 0.015x^2$

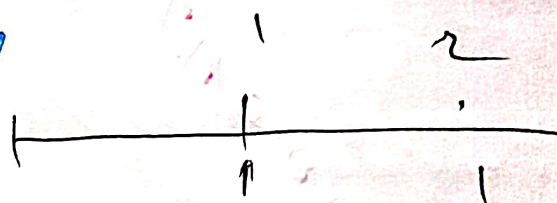


am

am

↓
 $PV = \frac{1}{(1+i)}$

$= v$



am

discount v v^2

am

Question-1

(a) what is meant by nominal rate of interest and force of interest?

Solution:

In many real life situations, the interest period is not a year, but rather some shorter period e.g. six months, a month, a week etc. That means, annual rate quoted with a compounded frequency. This is called nominal rate of interest.

$$i = \left(1 + \frac{j(m)}{m}\right)^m - 1$$

The effective annual rate of interest which accounts for compounding.

The force of interest represents the measure of the intensity of interest at a particular moment of time.

By definition,

$$\begin{aligned} \delta t &= \frac{a'(t)}{a(t)} \\ &= \ln(1+i) \end{aligned}$$

(b) Under usual notations, prove that

$$(i) \quad i = \left(1 + \frac{i(m)}{m}\right)^m - 1$$

$$(ii) \quad v = \frac{1}{1+i}$$

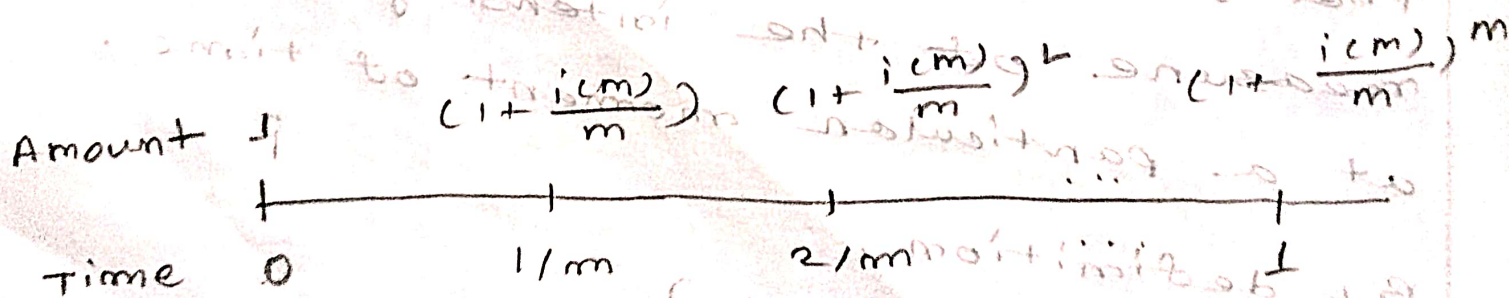
$$(iii) \quad i(m) = m \left\{ \left(1 + \frac{i(m)}{m}\right)^{\frac{1}{m}} - 1 \right\}$$

Solutions:

(i) The nominal rate of interest compounded m times per year is denoted by $i(m)$, so, the interest rate per period is:

$$\frac{i(m)}{m} + 1 = 1 + i$$

At each compounding step, the amount grows as follows:



The effective annual interest rate, i , is the total interest earned in one year,

$$i = \text{Final Amount} - \text{Initial Amount} \\ = \left(1 + \frac{im}{m}\right)^m - 1$$

$$\therefore i = \left(1 + \frac{im}{m}\right)^m - 1 \quad (\text{proved})$$

(ii)

The discount factor v represents the present value of 1 unit received at the end of one year,

After one year, the invested amount grows to 1,

$$v(1+i) = 1$$

$$\therefore v = \frac{1}{1+i}$$



(iii)

From part (i), the effective annual interest rate is:

$$i = \left(1 + \frac{icm}{m}\right)^m - 1$$

$$\Rightarrow 1+i = \left(1 + \frac{icm}{m}\right)^m$$

Now, we want to find the nominal interest rate compounded m times per year, denoted by icm .

the amount per compounding period

$$(1+i)^{1/m}$$

$$(1+i)^{1/m} = \left(1 + \frac{icm}{m}\right)^{m/m}$$

(iii)

The nominal rate $i(m)$ is compounded m times per year.

From part (i),

$$1 + i = \left(1 + \frac{i(m)}{m}\right)^m$$

We seek the nominal interest rate $i(m)$ that, when compounded m times per year, yields the same effective annual interest rate i .

The accumulated value after one year is:

$$\left(1 + \frac{i(m)}{m}\right)^m$$

Setting this equal to the effective annual accumulation:

$$\left(1 + \frac{i(m)}{m}\right)^n = \left(1 + \frac{i(m)}{m}\right)^m$$

$$\Rightarrow 1 + \frac{i(m)}{m} = \left(1 + \frac{i(m)}{m}\right)^{\frac{m}{n}}$$

$$\frac{i(m)}{m} = \left(1 + \frac{i(m)}{m}\right)^{\frac{m}{n}} - 1$$

$$\therefore i(m) = m \left[\left(1 + \frac{i(m)}{m}\right)^{\frac{m}{n}} - 1 \right]$$

(proved)

ced) Mention some practical applications of annuities and perpetuities.
Solution:

Application of Annuities

- ① Individual invest in annuities to receive a steady income stream after retirement.
- ② Monthly payments for home loans, car loans, or student loans follow an annuity structure.
- ③ Life insurance and pension plans give regular payouts like annuities.
- ④ Some college savings plans provide annuity like disbursements to cover tuition fees.
- ⑤ Rental payments made periodically follow an annuity structure.

Applications of Perpetuity

① Universities and charities use perpetuities to fund scholarships and operational cost indefinitely.

② If we own land on a building, rent can be collected forever.

③ Some roads and bridges charge tolls forever, making money like a perpetuity.

(d) Under usual notations, show algebraically that

(i) $a_{\infty} = \frac{1}{i}$

(ii) $s_{\infty} = \frac{c(1+i)^n - 1}{i}$

(iii) $s_{m+n} = s_m + (1+i)^m s_n$

(iv) $ia_{\infty} = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i-g}$

Solution

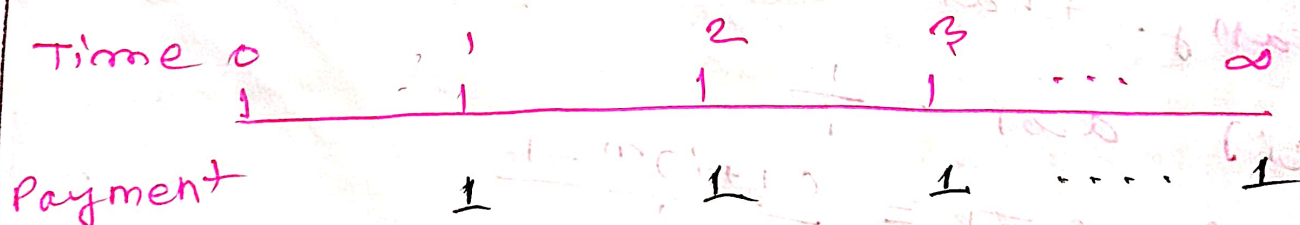
(i)

$$a_{\infty} = \lim_{n \rightarrow \infty} a_n$$

$$= \lim_{n \rightarrow \infty} \frac{1 - v^n}{1 - v}$$

$$= \frac{1}{1 - v} \quad \lim_{n \rightarrow \infty} v^n = 0 \text{ and } v > 0$$

Below is the time diagram that represents the perpetuity payments:



(iv)

The present value of an annuity with payments growing at a constant rate g is given by,

$$PV = \sum_{t=1}^n P_t (1+i)^{-t}$$

where,

$$P_1 = 1$$

$$P_2 = (1+g)$$

$$P_3 = (1+g)^2$$

$\vdots$

$$P_n = (1+g)^{n-1}$$

Thus the present value formula expands as,

$$PV = \sum_{t=0}^{n-1} (1+g)^t (1+i)^{-(t+1)}$$

$$= \sum_{t=0}^{n-1} \left(\frac{1+g}{1+i} \right)^t (1+i)^{-1}$$

This forms a geometric series with:

- First term: $\frac{1}{1+i}$
- common ratio: $r = \frac{1+g}{1+i}$
- number of terms: n

we know that, the sum of a finite geometric series is:

$$S_n = \frac{1 - r^n}{1 - r}$$

Applying this formula:

$$PV = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{1 - \frac{1+g}{1+i}}$$

$$= \frac{1}{1+i} \left[\frac{1 - \left(\frac{1+g}{1+i}\right)^n}{1 - \frac{1+g}{1+i}} \times \frac{1+i}{1-g} \right]$$

$$\therefore I \bar{a}_n = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i - g}$$

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2.

(c)

Given,

$$b(p) = \sqrt{1 + i^2 + 2ip^2}, \quad i > 0, \quad p > 0$$

we need to prove:

• $b(p) < 1 + ip$ for $0 < p < 1$

• $b(p) > 1 + ip$ for $p > 1$

Squaring both sides of $b(p)$

$$b(p)^2 = 1 + i^2 + 2ip^2$$

$$\text{and } (1 + ip)^2 = 1 + 2ip + i^2p^2$$

$$(1 + ip)^2 - b(p)^2 =$$

$$= 1 + 2ip + i^2p^2 - (1 + i^2 + 2ip^2)$$

$$= 1 + 2ip + i^2p^2 - 1 - i^2 - 2ip^2$$

$$= 2ip - 2ip^2$$

$$= 2ip(1 - p)$$

Now,

if $0 < p < 1$, then $1 - p > 0$

so, $2ip(1 - p) > 0$

This implies:

$$(1+p)^2 > bcp^2$$

$$\Rightarrow 1+p > bcp$$

$$\therefore bcp < 1+p \quad ; \quad -0 < p < 1$$

and,

if $p > 0$, then $1-p < 0 \Rightarrow 1 > p$

$$\text{so, } 2(1+p) < 0$$

this implies:

$$(1+p)^2 < bcp^2$$

$$\Rightarrow 1+p < bcp$$

$$\therefore bcp > 1+p \quad ; \quad p > 1$$

(removed)

1)

(a)

Principal: The initial amount of money we invest, before interest is applied.

Ex: If we deposit \$1000 in a bank, this is our principal.

4 (b)

Given,

- An annuity of k per year starts at age $x+m$.
- Annuity continues until the limiting age ω .
- Premiums paid from age x to $x+m-1$.
- Annuity paid from age $x+m$ until ω .

Cash flow diagram

	x	$x+1$	$x+2$	$x+m-1$	$x+n$	$x+n+1$
Age						
Premium	p	p	p	p	0	0
Annuity	0	0	0	0	k	k
Discount	1	v	v^2	v^{m-1}	v^n	v^{n+1}
Probability	1	${}_1p_x$	${}_2p_x$	${}_{n-1}p_x$	${}_np_x$	${}_{n+1}p_x$

Present value of premiums

Premiums form a term annuity due for n years:

$$\begin{aligned}
 PV_{\text{premiums}} &= p \ddot{a}_{x:n} \\
 &= p \sum_{j=0}^{n-1} v^j \cdot {}_j p_x
 \end{aligned}$$

Present value of Annuity

The annuity payments form a whole life annuity due, deferred by n years.

$$\begin{aligned}
 PV_{\text{annuity}} &= k \cdot {}_n| \ddot{a}_x \\
 &= k \cdot \sum_{t=n}^{\infty} v^t \cdot {}_t p_x
 \end{aligned}$$

Setting the present values equal:

$$P \cdot \ddot{a}_{x:\overline{n}|} = K \cdot n | \ddot{a}_x$$

$$\therefore P = K \cdot \frac{n | \ddot{a}_x}{\ddot{a}_{x:\overline{n}|}}$$

(2)

Age	40	41	...	54	55	...	64	65	66
Premium	P	$\frac{P}{2}$		P	$\frac{P}{2}$		$\frac{P}{2}$	0	0
Annuity	0	0	...	0	0	...	0	1	1

Premiums are paid from 40 to 64.

The annuity paid from 65 to 110.

present value of premiums

Full premium P from age 40 to 54
(15 years)

$$P.V \text{ full premiums} = P \cdot \ddot{a}_{40:\overline{15}|}$$

$$= P \sum_{k=0}^{14} v^k \cdot {}_k p_{40}$$

Reduced premium $\frac{P}{2}$ from age
55 to 64 (10 years)

$$PV \text{ half premium} = \frac{P}{2} \cdot 15 | \ddot{a}_{40} + \frac{P}{2} \sum_{k=15}^{24} v^k \cdot k P_{40}$$

$$PV \text{ premium} = \frac{P}{2} \cdot 15 | \ddot{a}_{40} + \frac{P}{2} \cdot 15 | \ddot{a}_{40}$$

Present value of annuity

$$PV \text{ annuity} = \sum_{t=25}^{69} v^t \cdot t P_x$$

$$= 25 | \ddot{a}_{40}$$

$$\therefore P = \frac{25 | \ddot{a}_{40}}{\ddot{a}_{40} | 15 + \frac{1}{2} 15 | \ddot{a}_{40}}$$

$$= \frac{25 | \ddot{a}_{40}}{\ddot{a}_{40} | 15 + \frac{1}{2} 15 | \ddot{a}_{40}}$$

1)

(e)

A loan is repaid by equal annual payments of 1 for n years. Each payment consists of two parts:

① Interest component

② Principal component

The fixed annual payment A is calculated as:

$$A = L \cdot \frac{i(1+i)^n}{(1+i)^n - 1} = 1$$

where,

L = loan amount

i = annual interest rate

n = number of payments.

Amortization schedule

Year	Payment (A)	Interest ($\text{Balance} \times i$)	Principal Repaid ($A - \text{Interest}$)	Loan balance LO.P
0	-	-	-	L
1	1	$L i$	$1 - L i$	$L - \text{principal}$
2	1	New balance $\times i$	$1 - \text{Interest}$	Updated balance
...	...	...	...	...
n	1	Last interest calculation	Remaining principal	0

At the end of n years, the loan is fully repaid.

(f) A sinking fund is a reserve account where funds are regularly deposited to repay a bond or debt at maturity. The amount deposited annually (S) is calculated as:

$$S = L \times \frac{j}{(1+j)^n - 1}$$

where:

L = loan amount

j = Annual interest rate of the sinking fund

n = total number of payments

Sinking Fund Schedule		Amortization Schedule	
Year	Interest (I)	Year	Payment (P)
0	-	0	-
1	-	1	-
2	-	2	-

interest
earned

Year	Payment Amount	Interest paid	Deposit in the sinking fund	sinking fund balance
------	----------------	---------------	-----------------------------	----------------------

0.00

0

1 $iL + S$

iL

S

S

2

$iL + S$

iL

S

previous value
+ S + previous
values

...

$iL + S$

iL

S

L

At the end of n years, the accumulated fund reaches the loan amount L , and the loan is repaid in full.

is repaid in full.

Year Deposit in the sinking fund (S) Interest earned (previous balance $\times i$) sinking fund balance

0

S

0

S

1

S

$S \times i$

previous value + S

2

...

S

Interest on loan + balance

L

n

$$X = 40$$

$$SA = 2000$$

$$n = 10$$

$$\text{Initial expense} = 20 + 5 + PV$$

$$PV = \sum_{t=0}^{n-1} \frac{1}{(1+i)^t} \times \frac{1}{(1+i)^t}$$

present value of paying

14 every year for 10 years

$$\text{Gross premium} = \frac{PV \text{ of death costs} + PV \text{ of expenses}}{10}$$

$$P = 1235.51$$

Age 25 36 ... 64 65 ...

$$PV = 1000 \sum_{t=0}^{44} \frac{1}{(1+i)^t} \frac{datt}{x}$$

$$1+i = (1 + \frac{0.112}{12})^{12} = 1.12$$

$$\Rightarrow i = (1 + \frac{0.112}{12})^{12} - 1$$

$$29 \sum_{t=0}^{44} \frac{1}{(1+i)^t} = 29 \times 11.900$$

$$s_m = \frac{(1+i)^n - 1}{i}$$

$$a_m = \frac{1 - v^n}{i}$$

$$= \frac{1 - (1+i)^{-n}}{i}$$

$$a_{m1} = \frac{1 - (1+i)^{-n}}{i}$$

$$= 11.33$$

$$a_{m2} = \frac{1 - (1+0.07)^{-10}}{1+i}$$

$$= 7.515$$

$$\text{PV of premiums} = 11.33 + 0.8 \times 7.515$$

$$= 12.346$$

$$\text{PV of expenses} = 10 + 10 \times 17.54$$

$$= 189.46$$

$$10 + 10 \times \left[\frac{1 - (1+i)^{-30}}{1+i} \right]$$

$$PV = 1000 \left(\frac{1 - (1 + 0.07)^{-45}}{0.07} \right)$$

$$= 13605.5$$

$$\text{Gross premium} = 13788.96$$

$$\text{Annual} = \frac{13788.96}{45} = 306.42$$

$$= 494.9$$

$$= 564.95$$

$$\frac{1 - (1 + 0.07)^{-45}}{0.07} = 11.6049$$

$$= 11.6049$$

$$1000 \times 11.6049 = 11604.9$$

$$1+i = \left(1 + \frac{i \text{ cm}}{m}\right)^m$$

$$A = P \frac{i(1+i)^m}{(1+i)^m - 1}$$

PV of death benefit

$$= PV \times \sum_{t=0}^{m-1} \frac{d_{m+t}}{1+i^t}$$

$$\sum_{t=0}^{m-1} \frac{1}{1+i^t}$$

PV of pay in, \$1000

$$= \sum_{t=0}^{m-1} v^t \times \frac{1000}{1+i^t}$$

$$v^t \times 1000$$

$$n \text{ pay } = n \times 1000$$

$$PV \text{ of } 11' = PV \times \frac{1000(1+i)^m - 1000}{1+i^m}$$

cost of PV system

prod

$$G.P = P.V.O.S + P.V.O.S$$

$$G.P = \frac{P.V.O.S}{10 + 10 \times 10 P.V}$$

$$V_1 = \frac{1 - V_m}{1 + 1}$$

23

(1) a b ab e ac be

000 100 010 110 001 101 011

young e = 0

51 den = 1

100 = 0

high = 2

100 = 0

a = 1

Department of Statistics and Data Science
Part IV B. Sc. (Honors) 3rd Tutorial Examination, 2023
Course Title: Actuarial Statistics
Course No. STAT 406

Total marks: 20

Total Time: 60 Minutes

Answer the following questions.

Q1. a) What is meant by nominal rate of interest and force of interest?

b) Under usual notations, prove that

i) $i = \left[1 + \frac{i^{(m)}}{m}\right]^m - 1$ ii) $v = \frac{1}{1+i}$ iii) $i^{(n)} = n\left\{\left[1 + \frac{i^{(m)}}{m}\right]^{\frac{1}{n}} - 1\right\}$

c) Mention some practical applications of annuities and perpetuities

d) Under usual notations, show algebraically that

i) $a_{\infty|} = \frac{1}{i}$

ii) $a_{n|} = \frac{(1+i)^n - 1}{i}$

iii) $s_{m+n|} = s_{m|} + (1+i)^n s_{n|}$

iv) $Ia_{n|} = \frac{1 - \left(\frac{1+g}{1+i}\right)^n}{i-g}$

e) What is a loan amortization schedule? Consider a loan which is being repaid by equal annual payments of 1 for n years. Construct an amortization schedule ✓

f) A loan of L is to be repaid by sinking fund method over n years. Find the (equal) periodic payment of the sinking fund. Also, construct sinking fund schedule ✓

Q2. a) Explain mathematically and verbally why the following expressions are true? ✓

i) ${}_nq_x = 1 - \frac{l_{x+n}}{l_x}$ ii) ${}_{m+n}p_x = ({}_mp_x)({}_np_{x+m})$

b) What is the difference between net premium and office premium?

c) Draw a cash flow diagram of an n -year term insurance of face value 1 on (x) . Derive the expression to find net single premium

d) An annuity on (x) provides k annually beginning at age $(x+n)$. Nothing is paid before n years. Annual premiums are payable for n years beginning at x . Draw a cash flow diagram and obtain the expression for annual premium, assuming limiting age ω

Net premium vs office premium

- The net premium is the amount to exactly cover the expected cost of future claims or benefits, excluding any additional expenses.
- The office premium is the actual amount charged to the policyholder. It includes additional loading for:
 - Administrative expenses.
 - commissions paid to agents.
 - contingencies and profit margins.

alive at age $x+m$, so subtracting it from 1 gives the proportion of individuals who have died before reaching $x+m$,

①

(c)

consider a loan repaid in equal annual payments of 1 for n years at an interest rate i . The schedule includes the following columns:

<u>Duration</u>	<u>Payment</u>	<u>Interest</u>	<u>Principal repaid</u>	<u>outstanding principal</u>
0				P
1	1	Pi	$1 - Pi$	$P - (1 - Pi)$
2	1	$(P - (1 - Pi))i$	$1 - (P - (1 - Pi))i$	$P - (1 - Pi) - (P - (1 - Pi))i$
$\vdots$	$\vdots$			
n	1			0

which in actuarial notation is:

$${}_n+m p_x = {}_m p_x \cdot {}_m p_{x+m}$$

This formula follows from the Markov property of the survival models, which states that future survival depends on the current age, not the past.

(GPT)

$${}_n q_x = 1 - {}_n p_x$$

$$= 1 - \frac{{}_n l_{x+n}}{{}_n l_x}$$

where,

${}_n q_x$ represents the probability that a person aged x dies within the next n years.

${}_n l_x$ denotes the number of people alive at age x .

This formula states that the probability of dying within n years is simply one minus the probability of surviving those n years.

The ratio $\frac{{}_n l_{x+n}}{{}_n l_x}$ represents the proportion of the original population still

Question-2
(a)

(i)

$${}_{m+n}p_x = {}_m p_x \cdot {}_n p_{x+m}$$

$$= \frac{l_{x+m}}{l_x} \times \frac{l_{x+m+n}}{l_{x+m}}$$

=

$${}_{m+n}p_x = \frac{l_{x+m+n}}{l_x}$$

$$= \frac{l_{x+m}}{l_x} \cdot \frac{l_{x+m+n}}{l_{x+m}}$$

$$= {}_m p_x \times {}_n p_{x+m}$$

${}_{m+n}p_x$ is the probability that a person aged x survives $m+n$ years.

This probability can be split into two independent survival periods:

(1) Survives the first m years, ${}_m p_x$

(2) Given survival to age $x+m$, survive another n years, ${}_n p_{x+m}$

Since survival probabilities are conditional

$$P(T_x > m+n) = P(T_x > m) \cdot P(T_{x+m} > n)$$

(i)

$${}_n|m q_x = {}_n p_x - {}_{n+m} p_x$$

hence,

${}_n|m q_x$ represents the probability that a person aged x survives n years but dies within the next m years.

This can be written using the survival probabilities:

$${}_n|m q_x = P(T_x > n) - P(T_x > n+m)$$

where, T_x is the future lifetime random variable,

${}_n p_x$ is the probability of surviving n years,

${}_{n+m} p_x$ is the " " " " $n+m$ years,

The difference represents that the probability of surviving n years but not surviving $n+m$ years,

$$\begin{aligned} {}_n|m q_x &= {}_n p_x \times (1 - {}_m p_{x+n}) \\ &= {}_n p_x - {}_n p_x \times {}_m p_{x+n} \\ &= {}_n p_x - {}_{n+m} p_x \end{aligned}$$

$$nPa = \frac{dx + m}{dx}$$

$$n+m \cdot Pa = \frac{dx + m + m}{dx}$$

$$\frac{dx + m - dx + (m + m)}{dx}$$

$$n/m \cdot Pa = 1 - n/m \cdot Pa$$

$$n/m \cdot Pa$$

$$n \cdot Pa + m$$

$$1 - n \cdot Pa \times m \cdot Pa + m$$

$$\frac{dx + m}{dx} \times \frac{dx + m + m}{dx}$$

$$\frac{dx + m}{dx} - \frac{dx + m}{dx} \quad (*)$$

$$dx + m = dm$$

$$dx + m = \frac{dm}{dx}$$

$$1 - \frac{dm}{dx} = \frac{dx - dm}{dx}$$

(*) Swift bank promises 3.5% interest compounded continuously. If Ken deposits \$3000 in the bank, what will his balance be four years later?

Soln:

For continuous compounding,

$$A(t) = P e^{rt \times 4}$$

$$= 3000 \times e^{0.035 \times 4}$$

$$= \$3450.82$$

(*)

$$A_1(5) = 10,000 \left(1 + \frac{0.06}{12} \right)^{12 \times 5}$$

$$= 12704.89$$

$$A_2(5) = A_1(5) \left(1 + \frac{0.075}{4} \right)^{4 \times 5}$$

$$= 19557.63$$