Incomplete Block Design

Incomplete Block Design (IBD)

In a design, if the number of plots within blocks is less than the number of treatments, then the blocks are said to be incomplete. If the treatments are allocated to the different blocks once and only once, then the resulting design is known as incomplete block design. The concept of incomplete block design was first devised by R. Yates in 1936.

Therefore, an incomplete block design is one having v treatments and b blocks each of size k $(k < v)$, such that each of the treatment is replicated r times and each pair of treatment occurs once and only once in the same Block, where $v, b,$ and k are the parameters of IBD and $k < v$.

Example:

In comparing efficiency of several level of fertilizer, large number of plots in a block may lead to increase the heterogeneity in respect of natural fertility. So, the incomplete block design may be used to reduce error variation.

Types of Incomplete Block Design

Depending upon the replication of the pairs of treatments, IBD is different types

- Balanced Incomplete Block Design (BIBD)
- Partially Balanced Incomplete Block Design (PBIBD)
- Youden Square Design (YSD)
- Lattice Design (LD)
- Cyclic Design (CD)

What is the necessity of IBD?

In certain experiment by using complete block designs, we may not able to run all the treatment combinations in each block. Situation like this usually occurs because of shortage of experimental apparatus of facilities or physical size of the block. This situations lead the experimenters to use IBD.

Applications of IBD

- By IBD, the difference between various blocks can be properly estimated.
- It ensures equal precisions of the estimates of all pairs of treatment effect.
- A large number of treatments can be compared with relatively small blocks.
- It is possible to estimate heterogeneity to a greater extent than is possible with randomized blocks and Latin square blocks by IBD.
- IBD may be used to reduce error variation.

Balanced Incomplete Block Design

If we have v treatments and b blocks each containing k experimental units and we replicate each treatment. So that it satisfy the following conditions

- Each treatment is replicated r times in design $(r < b)$.
- Each pair of treatment is replicated λ times $(\lambda \le r)$.

Then the design is called balanced incomplete block design. The quantities v, b, r, k & λ are called the parameters of the design and must be an integer.

Model for the BIBD

The observations obtained from BIBD can be explained by the model

$$y_{ijl} = \mu + \alpha_i + \beta_j + e_{ijl} \quad ; \quad i = 1(1)b \ , \ j = 1(1)v \ , l = n_{ij} = 0 \ or \ 1$$

Where, y_{ijl} is the l^{th} factor corresponding to the i^{th} block and j^{th} treatment., μ is the general mean effect, α_i is the effect due to the i^{th} block, β_j is the effect due to the j^{th} treatment, e_{ijl} is the random error component.

The number of observations obtained from this design can be arranged in rows and columns according to number of blocks and treatments as follows:

$$N = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b \times v}$$

$$Where, \quad n_{ij} = \begin{cases} 1 & if \ i^{th} \ treatment \ occurs \ in \ the \ i^{th} \ block \\ 0 & Otherwise \end{cases} \quad i = 1(1)b \ ; \ j = 1(1)v$$

By definition of BIBD,

$$\sum_{i=1}^{b} n_{ij} = N_{.j} = r \quad ; \quad \sum_{j=1}^{v} n_{ij} = N_{i.} = k \quad ; \quad \sum_{j=j'}^{v} n_{ij} n_{ij'} = \lambda$$

The matrix N is called the incidence matrix.

Advantages of BIBD

- Analysis of data is simple in BIBD
- All treatments are compared with equal precision
- By BIBD, we can conduct an experiment for large number of treatments in homogeneous blocks
- Unbiased estimate of treatment effects are readily available
- It is some efficient than RBD and it covers many complex situations where block size is less than the number of treatments under study.

Disadvantage of BIBD

- The analysis becomes complicated if the treatments are subject to the different error variance.
- There are two sources of error in this design.
- If there is missing observations, the analysis is more complicated.
- BIBD is not available for every number of treatment.

Use / Application of BIBD

- If the number of possible experimented plots per block is less than the number of treatments under study, then BIBD is used. This is the situation associated with the chemical, biological and physiological problems.
- If the number of treatments are so large than the number of experimental unit within a block, it fails to ensure the homogeneity of blocks for the given experimental unit. In such cases BIBD is employed to analyze the data. This is the situation associated with most of the agricultural field experiment.
- It is used for various trail experiment.

Why BIBD is Called Balanced?

In BIBD, each treatment is replicated at the same time. For this reason, the variance of the estimate of each treatment effect is same. Since each treatment effect has same variance, it is called Balanced.

Types of BIBD

There are two types of BIBD.

- Symmetric BIBD
- Asymmetric BIBD

Symmetric BIBD

A BIBD is said to be symmetric if the member of treatments is equal to the number of blocks, i.e., $b = v$ and also the number of replication is equal to the number of plots i.e. $r = k$. In this case, the incidence matrix N is a square matrix i.e.

$$N = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{v1} & n_{v2} & \cdots & n_{vv} \end{pmatrix}_{v\times v} = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1b} \\ n_{21} & n_{22} & \cdots & n_{2b} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bb} \end{pmatrix}_{b\times b}$$

Example

A BIBD is

Block	*Treatment*		
1	1	2	3
2	1	4	5
3	1	6	7
4	2	4	6
5	2	5	7
6	3	4	7
7	3	5	6

Show that the BIBD is symmetric.

Solution

The incidence matrix for the BIBD is given by

Treeatment (v) / Block (b)	1	2	3	4	5	6	7	$\sum_j n_{ij} = N_{i\cdot} = k$
1	1	1	1	0	0	0	0	3
2	1	0	0	1	1	0	0	3
3	1	0	0	0	0	1	1	3
4	0	1	0	1	0	1	0	3
5	0	1	0	0	1	0	1	3
6	0	0	1	1	0	0	1	3
7	0	0	1	0	1	1	0	3
$\sum_i n_{ij} = N_{\cdot j} = r$	3	3	3	3	3	3	3	

From the incidence matrix, we get,

$$b = v = 7 \quad and \quad r = k = 3 \qquad \lambda = \text{number of pairs of treatments occur in } b \text{ block} = 1$$

Since

Pairs	*no.*	*Pairs*	*no.*	*Pairs*	*no.*	*Pairs*	*no.*
$(1, 2)$	1	$(2, 3)$	1	$(3, 5)$	1	$(5, 6)$	1
$(1, 3)$	1	$(2, 4)$	1	$(3, 6)$	1	$(5, 7)$	1
$(1, 4)$	1	$(2, 5)$	1	$(3, 7)$	1	$(6, 7)$	1
$(1, 5)$	1	$(2, 6)$	1	$(4, 5)$	1		
$(1, 6)$	1	$(2, 7)$	1	$(4, 6)$	1		
$(1, 7)$	1	$(3, 4)$	1	$(4, 7)$	1		

So the design is a symmetric BIBD.

Asymmetric BIBD

A BIBD is called asymmetric BIBD if the number of treatments is not equal to the number of blocks i.e. $b \neq v$.

Resolvable Design

A BIBD which can be split up into r group such that each group contains a complete replicate of all the treatments is called resolvable design.

For example, let us consider the following BIBD with parameters $v=4,\ b=6,\ r=3,\ k=2\ and\ \lambda=1$

Blocks	*Treatments*		*Groups*
1	1	2	*First set*
2	3	4	
5	1	3	*Second set*
4	2	4	
5	1	4	*Third set*
6	2	3	

Here, $b=6$, blocks are divided into $r=3$ sets each of $\frac{b}{r}=\frac{6}{3}=2$ blocks.

Moreover, each set contains each of the treatment occurring once and only once. Also $\lambda=1$. Hence the above design is resolvable BIBD.

Affine Resolvable Design

A resolvable design is said to be affine resolvable if $b=r+v-1$ and only blocks for different sets have $\frac{k^2}{v}$ treatments common where $\frac{k^2}{v}$ is an integer.

For example, let us consider the resolvable design with parameters $v=4,\ b=6,\ r=3,\ k=2\ and\ \lambda=1$

$$Now, \qquad b=r+v-1 \qquad \Rightarrow \quad 6=3+4-1=6$$

Therefore, the condition $b=r+v-1$ is satisfied. Also $\frac{k^2}{v}=1$ (integer) and only one block different sets have only one treatment $(\lambda=1)$ common. Hence the design is affine resolvable.

Complementary Design

From a BIBD with parameters $(b,\ k,\ v,\ r,\ \lambda)$ is each block is replaced by another block containing those element with one omitted in the original block, then the design formed by the new blocks is a BIBD with parameters $(b,\ v-k,\ v,\ v-r,\ v-2r+1)$ and is called the complementary design of original BIBD.

Balanced Design

An experimental design is called a balanced design if the difference between every pair of treatment means or if all estimated elementary treatment contrasts has equal variance.

Connected Design

An incomplete block design in which all treatment contrasts are estimable, is called connected design.

State and Prove the Following Parametric Relation of BIBD.

A BIBD has parameters b, v, r, k and λ which are not independent as they satisfy the following conditions or relations:

1. $bk = vr$
2. $r(k-1) = \lambda(v-1)$
3. $r < \lambda$
4. $b \geq v \quad \Rightarrow \quad r \geq k$
5. $b \geq v + r - k$
6. $v > k$

Proof

1. Since BIBD consists of b blocks of size k each, total number of observations is bk. On the other hand, the design contains v varieties each replicated r times. So that the total number of observations vr. Hence $bk = vr$.

2. We know the incidence matrix is given by

$$N = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b\times v}$$

For BIBD, $n_{ij} = 1 \;\; or \;\; 0$ and $n_{ij}^2 = n_{ij} \;\; and \;\; so$

$$\sum_{i=1}^{b} n_{ij}^2 = \sum_{i=1}^{b} n_{ij} = r \quad ; \quad \sum_{j=1}^{v} n_{ij}^2 = \sum_{j=1}^{v} n_{ij} = k \quad ; \quad \sum_{\substack{i=1 \\ j\neq h}}^{v} n_{ij} n_{ih} = \lambda \quad ; \quad i = 1(1)b \quad ; \quad j = 1(1)v$$

$$Now, \qquad N'N = \begin{pmatrix} n_{11} & n_{21} & \cdots & n_{b1} \\ n_{12} & n_{22} & \cdots & n_{b2} \\ \vdots & \vdots & \cdots & \vdots \\ n_{1v} & n_{2v} & \cdots & n_{bv} \end{pmatrix}_{v\times b} \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b\times v}$$

$$= \begin{pmatrix} \sum_{i=1}^{b} n_{i1}^2 & \sum_{i=1}^{b} n_{i1}n_{i2} & \cdots & \sum_{i=1}^{b} n_{i1}n_{iv} \\ \sum_{i=1}^{b} n_{i2}n_{i1} & \sum_{i=1}^{b} n_{i2}^2 & \cdots & \sum_{i=1}^{b} n_{i2}n_{iv} \\ \vdots & \vdots & \cdots & \vdots \\ \sum_{i=1}^{b} n_{iv}n_{i1} & \sum_{i=1}^{b} n_{iv}n_{i2} & \cdots & \sum_{i=1}^{b} n_{iv}^2 \end{pmatrix}_{v\times v} = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix}_{v\times v} \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

Now, post multiplying (1) by $I = (1,\ 1,\ \cdots,\ 1)'$

$$N'N\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{v\times 1} = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix}_{v\times v} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{v\times 1}$$

$$\Rightarrow \qquad N'NI = \begin{bmatrix} r+\lambda(v-1) \\ r+\lambda(v-1) \\ \vdots \\ r+\lambda(v-1) \end{bmatrix}_{v\times 1} \quad \cdots \quad \cdots \quad \cdots \quad (2)$$

Again, multiplying the incidence matrix N and I, we get,

$$NI = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b\times v} \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{v\times 1} = \begin{bmatrix} N_1 \\ N_2 \\ \vdots \\ N_b \end{bmatrix}_{b\times 1} = \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix}_{b\times 1} \qquad \cdots \quad \cdots \quad \cdots \qquad (3)$$

Now, pre-multiplying NI by N' we get,

$$N'NI = \begin{pmatrix} n_{11} & n_{21} & \cdots & n_{b1} \\ n_{12} & n_{22} & \cdots & n_{b2} \\ \vdots & \vdots & \cdots & \vdots \\ n_{1v} & n_{2v} & \cdots & n_{bv} \end{pmatrix}_{v\times b} \begin{bmatrix} k \\ k \\ \vdots \\ k \end{bmatrix}_{b\times 1} = \begin{bmatrix} rk \\ rk \\ \vdots \\ rk \end{bmatrix}_{v\times 1} \qquad \cdots \quad \cdots \quad \cdots \qquad (4)$$

From (2) and (4) we get,

$$\begin{bmatrix} r+\lambda(v-1) \\ r+\lambda(v-1) \\ \vdots \\ r+\lambda(v-1) \end{bmatrix}_{v\times 1} = \begin{bmatrix} rk \\ rk \\ \vdots \\ rk \end{bmatrix}_{v\times 1}$$

$$\Rightarrow \quad r+\lambda(v-1) = rk$$

$$\Rightarrow \quad \lambda(v-1) = r(k-1) \qquad \text{(Proved)}$$

3. We know that

$$\lambda(v-1) = r(k-1) \qquad \cdots \quad \cdots \quad \cdots \qquad (1)$$

For IBD we know that

$$k < v \qquad \Rightarrow \quad (k-1) < (v-1) \qquad \cdots \quad \cdots \quad \cdots \qquad (2)$$

Comparing (1) and (2) we can write

$$r > \lambda \qquad \text{(Proved)}$$

4. $b \ge v \quad \Rightarrow \quad r \ge k$

We know, N be the incidence matrix and is given by

$$N = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b\times v}$$

And we know, $$N'N = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix}_{v\times v}$$

$$\therefore \quad |N'N| = \begin{vmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{vmatrix}_{v\times v} \qquad \cdots \quad \cdots \quad \cdots \qquad (1)$$

Subtracting first column from the other columns of (1) we get,

$$|N'N| = \begin{vmatrix} r & \lambda-r & \cdots & \lambda-r \\ \lambda & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & 0 & \cdots & r-\lambda \end{vmatrix}_{v\times v} \qquad \begin{array}{|l} c_2' = c_2 - c_1 \\ c_3' = c_3 - c_1 \\ \cdots \quad \cdots \\ c_v' = c_v - c_1 \end{array}$$

Now, adding last $v-1$ rows with the first row of $N'N$, we get,

$$|N'N| = \begin{vmatrix} r+\lambda(v-1) & 0 & \cdots & 0 \\ \lambda & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & 0 & \cdots & r-\lambda \end{vmatrix}_{v\times v} = \{r+\lambda(v-1)\}(r-\lambda)^{v-1} \neq 0 \qquad Since \quad r>\lambda$$

Thus $N'N$ is non-singular matrix of order $v\times v$ and hence rank of $N'N$ is

$$r(N'N) = v$$

But $r(N'N) = r(N'N) = r(N) = r(N') = v$. Again, since N is a $b\times v$ matrix, so $r(N)\leq b$

Thus, we have $\quad v\leq b \qquad\qquad (\text{Proved})$

Again, we know $vr = bk$

$$\therefore \qquad r\geq k \qquad Since \quad b\geq v \qquad (\text{Proved})$$

This inequality was derived and proved by Fisher in 1940 and hence is called Fisher's inequality.

5. $b\geq v+r-k$

We know that $b\geq v \;\; and \;\; r\geq k$ and for an incomplete block design we know $v>k$, so $(r-k)\geq 0$ and $(v-k)>0$.

$$\begin{aligned} \therefore \qquad & (r-k)(v-k)\geq 0 \\ \Rightarrow \qquad & vr \geq vk+kr-k^2 \qquad (Since \quad vr=bk) \\ \Rightarrow \qquad & b\geq v+r-k \qquad\qquad (\text{Proved}) \end{aligned}$$

Theorem: For a symmetric BIBD, any two blocks have λ treatments in common i.e., $r_{ii'}=\lambda$, where $r_{ii'}=\sum_j n_{ij}n_{i'j} \;\; ; \; i\neq i'=1(1)b$ and $\lambda=\sum_i n_{ij}n_{ij'} \;\; ; \; j\neq j'=1(1)v$.

Proof

We know, N be the incidence matrix and is given by

$$N = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b\times v}$$

$$NN' = \begin{pmatrix} n_{11} & n_{12} & \cdots & n_{1v} \\ n_{21} & n_{22} & \cdots & n_{2v} \\ \vdots & \vdots & \cdots & \vdots \\ n_{b1} & n_{b2} & \cdots & n_{bv} \end{pmatrix}_{b\times v} \begin{pmatrix} n_{11} & n_{21} & \cdots & n_{b1} \\ n_{12} & n_{22} & \cdots & n_{b2} \\ \vdots & \vdots & \cdots & \vdots \\ n_{1v} & n_{2v} & \cdots & n_{bv} \end{pmatrix}_{v\times b}$$

$$= \begin{pmatrix} k & r_{12} & \cdots & r_{1b} \\ r_{21} & k & \cdots & r_{2b} \\ \vdots & \vdots & \cdots & \vdots \\ r_{b1} & r_{b2} & \cdots & k \end{pmatrix}_{b\times b} \qquad \begin{bmatrix} Since & \sum_{j=1}^{v} n_{ij}^2 = k \\ and & \sum_{j=1}^{v} n_{ij}n_{i'j} = r_{ii'} \end{bmatrix} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (1)$$

Since $b=v$ for symmetric BIBD and $r=k$, therefore, N is a square matrix and $r(N)=v$ i.e. full rank. So N^{-1} exists. Thus $NN^{-1}=I$.

$$\therefore \quad NN' = NN'NN^{-1} = N(N'N)N^{-1} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (2)$$

Again, we know,

$$N'N = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix}_{v\times v} = \begin{pmatrix} r-\lambda & 0 & \cdots & 0 \\ 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r-\lambda \end{pmatrix}_{v\times v} + \begin{pmatrix} \lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & \lambda \end{pmatrix}_{v\times v}$$

$$= (r-\lambda)\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}_{v\times v} + \lambda\begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \cdots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}_{v\times v}$$

$$= (r-\lambda) I_{v\times v} + \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{v\times v} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times v}$$

Now, pre-multiplying by N we get,

$$N(N'N) = (r-\lambda)N + \lambda N \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{v\times v} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}_{1\times v} \qquad (Since \quad NI = N)$$

$$= (r-\lambda)N + \lambda \begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} \qquad \left(Since \quad N\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} k \\ k \\ \vdots \\ k \end{pmatrix} \right)$$

$$= (r-\lambda)N + \lambda k \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix}$$

Again, post-multiplying by N^{-1}, we get,

$$N(N'N)N^{-1} = (r-\lambda)NN^{-1} + \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} kN^{-1}$$

$$= (r-\lambda)I + \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} r & r & \cdots & r \end{pmatrix} N^{-1} \qquad (Since \quad r = k)$$

$$= (r-\lambda)I + \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} NN^{-1} \qquad \left(Since \quad \begin{pmatrix} r & r & \cdots & r \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} N \right)$$

$$= (r-\lambda)I + \lambda \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \cdots & 1 \end{pmatrix} I \qquad = \begin{pmatrix} r-\lambda & 0 & \cdots & 0 \\ 0 & r-\lambda & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & r-\lambda \end{pmatrix}_{v\times v} + \begin{pmatrix} \lambda & \lambda & \cdots & \lambda \\ \lambda & \lambda & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & \lambda \end{pmatrix}_{v\times v}$$

$$= \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix}_{v\times v} = N'N \qquad \cdots \quad \cdots \quad \cdots \quad (3)$$

Computing (1), (2) and (3) we get,

$$\begin{pmatrix} k & r_{12} & \cdots & r_{1b} \\ r_{21} & k & \cdots & r_{2b} \\ \vdots & \vdots & \cdots & \vdots \\ r_{b1} & r_{b2} & \cdots & k \end{pmatrix} = \begin{pmatrix} r & \lambda & \cdots & \lambda \\ \lambda & r & \cdots & \lambda \\ \vdots & \vdots & \cdots & \vdots \\ \lambda & \lambda & \cdots & r \end{pmatrix}$$

Hence, we get, $r = k$ and $r_{12} = \lambda,\ r_{13} = \lambda,\ \cdots,\ r_{1b} = \lambda \quad and\ so\ on.$

In general, $r_{ii'} = \lambda$ $\qquad (\text{Proved})$

Theorem: A necessary condition for the existence of a symmetric BIBD with even number of treatments is that $(r-\lambda)$ is perfect square.

Proof

In a symmetric BIBD, we have, $b = v,\ r = k$ and incidence matrix N is a square matrix and we know,

$$|N'N| = \{r + \lambda(v-1)\}(r-\lambda)^{v-1} \qquad = \{r + r(k-1)\}(r-\lambda)^{v-1} \qquad [Since\ \ \lambda(v-1) = r(k-1)]$$

$$= rk(r-\lambda)^{v-1}$$

$$\Rightarrow \qquad |N'||N| = r^2(r-\lambda)^{v-1} \qquad [Since\ \ r = k]$$

$$\Rightarrow \qquad |N|^2 = r^2(r-\lambda)^{v-1} \qquad [Since\ \ N\ \ is\ \ symmetric]$$

$$\Rightarrow \qquad |N| = \pm r(r-\lambda)^{\frac{v-1}{2}}$$

Which is an integer, since all the elements of incidence matrix are integer (because the elements of N of BIBD is either 0 or 1) and r and λ both are integers. Therefore, $(r-\lambda)$ must be a perfect square.

Theorem: For a BIBD, with parameters $b,\ r,\ v,\ k\ \ and\ \ \lambda$ where b is divisible by r then show that $b \geq v + r - 1$

Proof

Since b is divisible by r, then $\dfrac{b}{r}$ is an integer, say n.

$$\therefore \quad b/r = n \qquad \Rightarrow \qquad b = nr \qquad \cdots \qquad \cdots \qquad \cdots \qquad (1)$$

Again, we know for BIBD

$$bk = vr \qquad \Rightarrow \qquad nrk = vr \qquad [by\ \ (1)]$$

$$\Rightarrow \qquad v = nk \qquad \cdots \qquad \cdots \qquad \cdots \qquad (2)$$

Also for BIBD,

$$\lambda(v-1) = r(k-1) \qquad \cdots \qquad \cdots \qquad \cdots \qquad (3)$$

$$\Rightarrow \qquad r = \lambda\frac{(v-1)}{(k-1)} \qquad = \lambda\frac{(nk-1)}{(k-1)}$$

$$= \frac{\lambda nk - n\lambda + n\lambda - \lambda}{(k-1)}$$

$$= n\lambda + \frac{\lambda(n-1)}{(k-1)} \qquad = \text{Positive integer} \qquad (Since\ \ n > 1,\ k > 1)$$

Since $b \geq v$, so $b \geq v + r - 1$ if it is not hold.

Let $b < v + r - 1 \quad \Rightarrow \quad nr < v + r - 1 \qquad [by \ (1)]$

$$\Rightarrow \quad r(n-1) < v - 1$$

$$\Rightarrow \quad r(n-1) < \frac{r}{\lambda}(k-1) \qquad [by \ (3)]$$

$$\Rightarrow \quad \frac{\lambda(n-1)}{k-1} < 1$$

which is impossible, so $b < v + r - 1$ cannot be true. So $b \geq v + r - 1$ (Proved)

Analysis of Data Obtained BIBD

There are two types of analysis of BIBD

i) Intra-block Analysis

ii) Inter-block Analysis

Intra-Block Analysis

When blocks are fixed, i.e. we want to find the treatment comparison within blocks, then the analysis of BIBD is known as intra-block analysis.

Inter-Block Analysis

When we want to find the treatment comparison between blocks, then the analysis of BIBD is known as inter-block analysis.

Intra-Block Analysis of BIBD

The usual model for the intra block analysis of BIBD is

$$y_{ijl} = \mu + \alpha_i + \beta_j + e_{ijl} \quad ; \quad i = 1(1)b \ ; \ j = 1(1)v \ ; \ l = n_{ij} = 0 \ or \ 1$$

where, y_{ijl} is the l^{th} observation of the j^{th} treatment in the i^{th} block, μ is the general mean effect, α_i is the i^{th} block effect, β_j is the j^{th} treatment effect and e_{ijl} is the random error component.

Assumption

i) $\mu, \ \alpha_i, \ \beta_j$ are unknown parameters

ii) $e_{ijl} \sim NID(0, \sigma^2)$

iii) There is no interaction between blocks and treatments

Estimation of Parameters

Applying least square method, we get the following normal equation

$$E = \sum_i \sum_j \sum_l e_{ijl}^2 = \sum_i \sum_j \sum_l \left(y_{ijl} - \mu - \alpha_i - \beta_j\right)^2$$

$$\therefore \quad \frac{\partial E}{\partial \mu} = 0 \quad \Rightarrow \quad y_{...} = \sum_i \sum_j n_{ij}\mu + \sum_i N_{i.}\alpha_i + \sum_j \beta_j N_{.j}$$

$$\Rightarrow \quad y_{...} = bk\hat{\mu} + k\sum_i \hat{\alpha}_i + r\sum_j \beta_j \qquad \cdots \qquad \cdots \qquad \cdots \qquad (1)$$

$$\frac{\partial E}{\partial \alpha_i} = 0 \quad \Rightarrow \quad y_{i..} = k\hat{\mu} + k\hat{\alpha}_i + \sum_j n_{ij}\beta_j \qquad \cdots \qquad \cdots \qquad \cdots \qquad (2)$$

$$\frac{\partial E}{\partial \beta_j} = 0 \quad \Rightarrow \quad y_{.j.} = r\hat{\mu} + \sum_i n_{ij}\hat{\alpha}_i + r\beta_j \qquad \cdots \qquad \cdots \qquad \cdots \qquad (3)$$

For getting the unique solution, we have to put the restrictions $\sum_i \hat{\alpha}_i = \sum_j \beta_j = 0$.

From (1) we get, $\hat{\mu} = \frac{1}{bk} y_{...} = \overline{y}_{...}$

From (2) we get, $\hat{\alpha}_i = \frac{1}{k}\left(y_{i..} - k\hat{\mu} - \sum_j n_{ij}\hat{\beta}_j \right)$

Putting the value of $\hat{\alpha}_i$ in (3), we get,

$$y_{.j.} = r\hat{\mu} + \sum_i n_{ij}\left\{ \frac{1}{k}\left(y_{i..} - k\hat{\mu} - \sum_j n_{ij}\beta_j \right) \right\} + r\beta_j$$

$$= r\hat{\mu} + \frac{1}{k}\sum_i n_{ij} y_{i..} - r\hat{\mu} - \frac{1}{k}\sum_i n_{ij}\sum_j n_{ij}\beta_j + r\beta_j$$

$$\Rightarrow \quad y_{.j.} - \frac{1}{k}\sum_i n_{ij} y_{i..} = r\beta_j - \frac{1}{k}\sum_i n_{ij}\sum_j n_{ij}\beta_j$$

$$\Rightarrow \quad Q_j = r\beta_j - \frac{1}{k}\left[\sum_i n_{ij}^2\beta_j + \sum_i \sum_{j \neq s = 1} n_{ij} n_{is}\beta_s \right]$$

$$= r\beta_j - \frac{1}{k}\sum_i n_{ij}^2 \beta_j - \frac{1}{k}\lambda \sum_{\substack{s=1 \\ s \neq j}} \beta_s$$

$$= r\beta_j - \frac{r}{k}\beta_j - \frac{\lambda}{k}\left(\sum_{s=1} \beta_s - \beta_j \right) \qquad \left(Since \quad \sum_{s=1} \beta_s = 0 \right)$$

$$= \beta_j \left(r - \frac{r}{k} + \frac{\lambda}{k} \right)$$

$$\Rightarrow \quad Q_j = \beta_j \frac{v\lambda}{k} \qquad \left(Since \;\; r(k-1) = \lambda(v-1) \right)$$

$$\therefore \quad \beta_j = \frac{k}{v\lambda} Q_j \quad = \frac{Q_j}{r\frac{v\lambda}{kr}} \quad = \frac{Q_j}{rE}$$

Where $E = \frac{\lambda v}{kr}$ is the efficiency factor of BIBD. β_j is the intra block estimate of j^{th} treatment effect.

Now, sum of square due to estimate,

$$S_1^2 = \hat{\mu} y_{...} + \sum \hat{\alpha}_i y_{i..} + \sum \hat{\beta}_i y_{.j.}$$

$$= \hat{\mu} y_{...} + \sum \frac{1}{k}\left(y_{i..} - k\hat{\mu} - \sum_j n_{ij}\hat{\beta}_j \right) y_{i..} + \sum \hat{\beta}_j y_{.j.}$$

$$= \hat{\mu} y_{...} + \frac{1}{k}\sum y_{i..}^2 - \hat{\mu} y_{...} - \frac{1}{k}\sum\sum n_{ij}\hat{\beta}_j y_{i..} + \sum \hat{\beta}_j y_{.j.}$$

$$= \frac{1}{k}\sum y_{i..}^2 + \sum \hat{\beta}_j \left(y_{.j.} - \frac{1}{k}\sum n_{ij} y_{i..} \right)$$

$$= \frac{\sum y_{i..}^2}{k} + \sum \frac{Q_j}{rE} Q_j$$

$$= \frac{\sum y_{i..}^2}{k} + \sum \frac{Q_j^2}{rE} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (4)$$

Error sum of square,

$$ESS = \sum\sum\sum y_{ijl}^2 - SS(due \;\; to \;\; estimate) \;\; = \sum\sum\sum y_{ijl}^2 - \frac{\sum y_{i..}^2}{k} - \sum \frac{Q_j^2}{rE} \quad \cdots \quad \cdots \quad \cdots \quad (5)$$

The main object of BIBD is to test $H_0 : \beta_j = 0$.

Under $H_0 : \beta_j = 0$, the model be $y_{ijl} = \mu + \alpha_i + e_{ijl}$

Then using least square method we have,

$$\hat{\mu} = \bar{y}_{...} \qquad and \qquad \hat{\hat{\alpha}}_i = \frac{1}{k}\left(y_{i..} - k\hat{\mu}\right)$$

Now, the sum of square due to estimate under reduced model is

$$S_2^2 = \hat{\mu} y_{...} + \sum \hat{\hat{\alpha}}_i y_{i..}$$

$$= \hat{\mu} y_{...} + \sum \frac{1}{k}\left(y_{i..} - k\hat{\mu}\right) y_{i..} \quad = \frac{\sum y_{i..}^2}{k} \qquad \cdots \qquad \cdots \qquad \cdots \qquad (6)$$

So, Adjusted treatment sum of square is

$$S_3^2 = S_1^2 - S_2^2 \qquad = \frac{\sum Q_j^2}{rE}$$

$$SS\left(Unadjusted \;\; block\right) = \frac{1}{k}\sum y_{i..}^2 - CT \qquad where, \;\; CT = \frac{y_{...}^2}{bk}$$

ANOVA Table

S. V.	$d.f.$	SS	MSS	F
Block (Unadjusted)	$b-1$	$\frac{\sum y_{i..}^2}{k} - CT$	s_1	$F_1 = \frac{s_1}{s_3}$
Treatment (adjusted)	$v-1$	$\frac{\sum Q_j^2}{rE}$	s_2	$F_2 = \frac{s_2}{s_3}$
Intra Block Error	$bk-b-v+1$	$\sum\sum\sum y_{ijl}^2 - \frac{\sum y_{i..}^2}{k} - \frac{\sum Q_j^2}{rE}$	s_3	
Total	$bk-1$			

Now we want to test the hypothesis

$$H_0 : \beta_j = 0 \qquad against \qquad H_1 : \beta_j \neq 0$$

The test statistic is $F_2 = \frac{s_2}{s_3} \sim F_{\alpha,\,(v-1),\,(bk-b-v+1)}$

If $F_2 > F_{\alpha,\,(v-1),\,(bk-b-v+1)}$, we can reject H_0.

If $H_0 : \beta_j = 0$ is rejected then we can test

$$H_0 : \beta_j = \beta_{j'} \qquad i.e. \qquad H_0 : \beta_j - \beta_{j'} = 0 \qquad ; \qquad j \neq j' = 1(1)v$$

$$against \qquad H_0 : \beta_j - \beta_{j'} \neq 0$$

Test statistic, $$t = \frac{\left|\hat{\beta}_j - \hat{\beta}_{j'}\right|}{SE\left(\hat{\beta}_j - \hat{\beta}_{j'}\right)}$$

Here, $$\hat{\beta}_j - \hat{\beta}_{j'} = \frac{1}{rE}\left(Q_j - Q_{j'}\right)$$

$$\therefore \quad V\left(\hat{\beta}_j - \hat{\beta}_{j'}\right) = \frac{2MSE}{rE}$$

$$\therefore \quad t = \frac{\left|\hat{\beta}_j - \hat{\beta}_{j'}\right|}{\sqrt{\frac{2MSE}{rE}}} \sim t_{\alpha/2,\, bk-b-k+1}$$

If $t \geq t_{\alpha/2,\, bk-b-k+1}$, we can reject $H_0 : \beta_j = \beta_{j'}$.

Inter Block Analysis

There are two types of block

1. Treatment Block
2. Unit Block

If different set of treatments occur in different blocks in a BIBD, then a set of treatment occurring in a block will reflect the block effect. The treatments are selected at random in a block. The data can be analyzed by considering block effect as random. The corresponding analysis is called analysis with recovery inter block in coefficient. The linear model for inter block analysis of BIBD is given by

$$y_{ijl} = \mu + \alpha_i + \beta_j + e_{ijl} \quad ; \quad i = 1(1)b \;\; ; \;\; j = 1(1)v \;\; ; \;\; l = n_{ij} = 1 \;\; or \;\; 1$$

where, y_{ijl} is the l^{th} observation of the j^{th} treatment in the i^{th} block, μ is the general mean effect, α_i is the random effect of i^{th} block, β_j is the fixed effect of j^{th} treatment and e_{ijl} is the random error component.

Assumption

i) $\mu, \; \beta_j$ are unknown parameters and fixed

ii) Block effect α_i is random

iii) $\alpha_i \sim NID\left(0, \, \sigma_\alpha^2\right)$

iv) $e_{ijl} \sim NID\left(0, \, \sigma^2\right)$

v) $E\left(\alpha_i, \, e_{ijl}\right) = 0$

vi) $\sum \beta_j = 0$

Estimation of Parameters

The total field of i^{th} block is

$$y_{i\cdot\cdot} = k\mu + k\alpha_i + \sum_j n_{ij}\beta_j + e_{i\cdot\cdot}$$

$$\therefore \quad V\left(y_{i\cdot\cdot}\right) = V\left(k\mu + k\alpha_i + \sum_j n_{ij}\beta_j + e_{i\cdot\cdot}\right) = k^2\alpha_\alpha^2 + k\sigma^2$$

Therefore, the analysis can be performed by considering both α_i and e_{ijl} as error and the block total as observation. In that case we have to estimate the treatment effect by minimizing the ESS.

$$Q = ESS = \sum_i \left(y_{i\cdot\cdot} - k\mu - \sum_j n_{ij}\beta_j\right)^2 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

Now partially differentiating with respect to $\mu \;\; and \;\; \beta_j$ we get,

$$\frac{\partial Q}{\partial \mu} = 0 \quad \Rightarrow \quad 2\sum_i \left(y_{i\cdot\cdot} - k\mu - \sum_j n_{ij}\beta_j\right)(-k) = 0$$

$$\Rightarrow \quad y_{\cdots} - bk\mu - r\sum_j \beta_j = 0 \qquad \left[Since \;\; \sum \beta_j = 0\right]$$

$$\Rightarrow \quad \mu = \overline{y}_{\cdots}$$

$$\frac{\partial Q}{\partial \beta_j} = 0 \quad \Rightarrow \quad -2\sum_i \left(y_{i\cdot\cdot} - k\hat{\mu} - \sum_j n_{ij}\beta_j \right) n_{ij} = 0$$

$$\Rightarrow \quad \sum_i n_{ij} y_{i\cdot\cdot} - rk\hat{\mu} - \sum_j n_{ij}^2 \beta_j - \sum_i \sum_{j \neq s = 1} n_{ij} n_{is} \beta_s = 0$$

$$\Rightarrow \quad \sum_i n_{ij} y_{i\cdot\cdot} = rk\hat{\mu} + r\beta_j + \lambda \sum_{j \neq s = 1} \beta_s$$

$$\Rightarrow \quad \sum_i n_{ij} y_{i\cdot\cdot} = rk\hat{\mu} + r\beta_j + \lambda \sum_{s=1} \beta_s - \lambda\beta_j \qquad \left[Since \ \sum_{s=1} \beta_s = 0 \right]$$

$$\Rightarrow \quad \sum_i n_{ij} y_{i\cdot\cdot} = rk\hat{\mu} + r\beta_j - \lambda\beta_j$$

$$\Rightarrow \quad \hat{\beta}_j = \frac{\sum_i n_{ij} y_{i\cdot\cdot} - rk\hat{\mu}}{(r-\lambda)} = \frac{\sum_i n_{ij} y_{i\cdot\cdot} - rk\frac{y_{\cdots}}{bk}}{(r-\lambda)}$$

$$\Rightarrow \quad \hat{\beta}_j = \frac{kP_i}{(r-\lambda)} \qquad \left[Where, \ P_i = \frac{\sum_i n_{ij} y_{i\cdot\cdot}}{k} - r\frac{y_{\cdots}}{bk} \right]$$

Therefore, $SS(adjusted \ \ treatment) = \dfrac{k}{r-\lambda}\sum P_j^2$

$$SS(Unadjusted \ \ estimate) = \frac{1}{k}\sum y_{i\cdot\cdot}^2 + \frac{k}{r-\lambda}\sum P_j^2$$

ANOVA Table:

S. V.	$d.f.$	SS	MSS	F
Treatment (adjusted)	$v-1$	$S_1 = \frac{k}{r-\lambda}\sum P_j^2$	s_1	$F = \frac{s_1}{s_2}$
Inter Block Error	$b-v$	$S_2 = S_3 - S_1$	s_2	
Total	$b-1$	$S_3 = \frac{1}{k}\sum y_{i\cdot\cdot}^2 - CT$		

If $F \geq F_{\alpha.(v-1),(b-v)}$ then we reject $H_0 : \beta_j = 0$.

If $H_0 : \beta_j = 0$ is rejected, then we can test

$$H_0 : \beta_j = \beta_{j'} \qquad ; \qquad j \neq J' = 1(1)v$$

The test statistic $\quad t = \dfrac{\left|\hat{\beta}_j - \hat{\beta}_{j'}\right|}{SE\left(\hat{\beta}_j - \hat{\beta}_{j'}\right)}$

Here, $\quad \hat{\beta}_j - \hat{\beta}_{j'} = \dfrac{k}{r-\lambda}\left(P_j - P_{j'}\right)$

$$\Rightarrow \quad V\left(\hat{\beta}_j - \hat{\beta}_{j'}\right) = \frac{2k\left(\sigma^2 + \sigma_\alpha^2\right)}{r-\lambda}$$

$$\therefore \quad t = \frac{\left|\frac{k}{r-\lambda}\left(P_j - P_{j'}\right)\right|}{\sqrt{\frac{2k\left(\sigma^2 + \sigma_\alpha^2\right)}{r-\lambda}}} \quad \sim \quad t_{\alpha/2,\, b-v}$$

If $t \geq t_{\alpha/2,\, b-v}$, we reject $H_0 : \beta_j = \beta_{j'}$.

Distinction Between Intra-Block and Inter-Block Analysis of BIBD

In intra block analysis, the blocks are not selected randomly; as a result the block effect is fixed effect. On the other hand, in inter block analysis, the experimental block are selected randomly to allocate the treatments. As a result the block effect is random effect.

Importance of Inter-Block Analysis in BIBD

If the treatments are allocated randomly to a selected block (fixed) then the result of the experiment may be less efficient if the experimental block is less efficient. For removing this difficulty experimental blocks are selected randomly to allocate the treatments. As a result block effect is random effect and in inter-block analysis, block effect is considered as random effect.

So for more efficient of the experiment, when block is less efficient, inter block analysis is efficient.

Varietal Trial

Varietal trials are primarily used in agricultural experiments. The object of these trials is to select a few varieties which are better than the rest in respect of some economic character.

If the number of varieties is small, ordinary randomized block designs are sometimes Latin Square Designs can be used for the trials. But if the number of varieties in such trials is large, then adoption of RBD may increase error variance due to large block size. Therefore, when the number of varieties is large then RBD or LSD are not suitable.

Efficiency of Incomplete Block Design Relative to RBD

For RBD with r replications, the variance of the contrast $\hat{\beta}_j - \hat{\beta}_{j'}$ is $\frac{2\sigma_R^2}{r}$ and that of corresponding Incomplete Block Design is $\frac{2\sigma^2}{rE}$, where σ_k^2 is the error variance of RBD. So the efficiency of the symmetric in complete block design relative to RBD is

$$\frac{2\sigma_R^2 / r}{2\sigma^2 / rE} = E.\frac{\sigma_R^2}{\sigma^2}$$

Here E is the efficiency factor.

We know,

$$\begin{aligned} E &= \frac{\lambda v}{rk} = \frac{\lambda v - \lambda + \lambda}{rk + r - r} \\ &= \frac{\lambda(v-1)+\lambda}{r(k-1)+r} \quad \text{since } \lambda(v-1) = r(k-1) \\ &= \frac{r(k-1)+\lambda}{r(k-1)+r} < 1 \quad as \quad \lambda < r. \end{aligned}$$

So, symmetric incomplete block design is more efficient. $E.\frac{\sigma_R^2}{\sigma^2}$ must be greater than 1 which can happen only when $E < 1$. This means $\frac{\sigma_R^2}{\sigma^2}$ should be greater than 1.

Combined Intra and Inter Block Design

We have the model

$$y_{ijl} = \mu + \alpha_i + \beta_j + e_{ijl} \quad ; \quad i = 1(1)b,\; j = 1(1)v,\; l = n_{ij} = 0 \text{ or } 1$$

We know that the intra block variance is σ^2 and that of inter block is $\sigma^2 + k\,\sigma_\alpha^2$.

Let us suppose, $w = \frac{1}{\sigma^2}$ and $w_1 = \frac{1}{\sigma^2 + k\,\sigma_\alpha^2}$. So that the weighted sum of squares due to error is

$$S = w\sum_i \sum_j \sum_l \left(y_{ijl} - \hat{\mu} - \hat{\alpha}_i - \hat{\beta}_j \right)^2 + \frac{w_1}{k} \sum_i \left(y_{i..} - k\hat{\mu} - \sum_j n_{ij}\hat{\beta}_j \right)^2$$

Though minimizing S we will get the combined intra and inter block estimate of β_j.

Thus $\frac{\delta s}{\delta \beta_j} = 0$ gives,

$$w\left[y_{.j'.} - r\hat{\mu} - \sum n_{ij}\alpha_i - r\hat{\beta}_j \right] + \frac{w_1}{k}\left[\sum n_{ij} y_{ij} - rk\hat{\mu} - \sum_i n_{ij} \sum_j n_{ij}\hat{\beta}_j \right] = 0 \quad \cdots \quad \cdots \quad \cdots \quad (1)$$

From the block analysis we know that $\hat{\alpha}_i = \frac{1}{k} y_{i..} - \hat{\mu} - \frac{1}{k}\sum_j n_{ij}\hat{\beta}_j$

On substituting the values of α_i in (1) simplifying we get,

$$wQ_j + w_1 p_j = \left[wr\,E + \frac{w_1}{k}(r - \lambda) \right]\hat{\beta}_j$$

$$\therefore\; \hat{\beta}_j = \frac{Q_j + \rho\, p_j}{rE + \frac{\rho}{k}(r - \lambda)}, \quad \text{where } \rho = \frac{w_1}{w}$$

This estimate of β_j can also be obtained from the weighted mean of the estimates obtained from intra and inter block analysis.

We know,

$$V\left(\hat{\beta}_j\right)_{\text{intra}} = \frac{(v-1)\sigma^2}{v\,r\,E} \quad \textit{and}$$

$$V\left(\hat{\beta}_j\right)_{\text{intra}} = \frac{(v-1)\left(\sigma^2 + k\sigma_\alpha^2\right)}{\frac{v}{k}(r - \lambda)}$$

Let,

$$w = \frac{v\,r\,E}{(v-1)\sigma^2} \quad \text{and} \quad w_1 = \frac{v(r-\lambda)}{k(v-1)\left(\sigma^2 + k\sigma_\alpha^2\right)}$$

$$\therefore\; w + w_1 = \frac{v}{v-1}\left[\frac{rE}{\sigma^2} + \frac{r - \lambda}{k\left(\sigma^2 + k\sigma_\alpha^2\right)} \right]$$

$$= \frac{wv}{v-1}\left[rE + \frac{\rho}{k}(r - \lambda) \right]$$

So the combined intra and inter block estimate of β_j is

$$\hat{\beta}_j = \frac{1}{w + w_1}\left[\frac{Q_j}{rE} \cdot \frac{v\,r\,E}{(v-1)\sigma^2} + \frac{k\,p_j}{(r-\lambda)} \cdot \frac{v(r-\lambda)}{k(v-1)\left(\sigma^2 + k\sigma_\alpha^2\right)} \right]$$

Putting the value of $w+w_1$ and on simplifying we get,

$$\hat{\beta}_j = \frac{Q_j + \rho\, p_j}{r\,E + \frac{\rho}{k}(r-\lambda)}$$

In practical situation, w and w_1 are unknown. So that ρ is unknown. So that we have to estimate w and w_1.

We have,
$$\hat{\beta}_{j(\text{intra and inter})} = \frac{Q_j + \rho\, p_j}{r\,E + \frac{\rho}{k}(r-\lambda)} = \frac{w\hat{\beta}_{j(\text{intra})} + w_1\hat{\beta}_{j(\text{inter})}}{w+w_1}$$

Here $\hat{\beta}_j$ depends on w and w_1. Since w and w_1 are unknown, so that $\hat{\beta}_j$ is biased estimate.

After reducing bias, the estimate of $\hat{\beta}_j$ can be obtained following a theorem due to Mier (1953) as follows:

$$\hat{\beta}_j = \frac{\hat{w}\hat{\beta}_{j(\text{intra})} + \hat{w}_1\hat{\beta}_{j(\text{inter})}}{\hat{w}+\hat{w}_1} - \sum \frac{1}{f_i}\left[\frac{\delta^2 R}{\delta x_i^2}\right] \qquad \text{all } x_i = 1$$

$$where,\ R = \frac{\hat{w}\hat{\beta}_{j(\text{intra})} + \hat{w}_1\hat{\beta}_{j(\text{inter})}}{\hat{w}+\hat{w}_1} \quad ; \quad x_1 = \frac{\hat{w}}{w} \quad , \quad x_2 = \frac{\hat{w}}{w}$$

Construction of BIBD

- Prime number $P = 2,3,5,7,11,13,17,19,23,\cdots$.
- The remainder R, after dividing a positive number N by P, can be written as equal to N i.e.

$$R = N \bmod P$$

where N is positive number, P is any prime number, R is remainder.

$$For, \quad N = 19,\ P = 7 \quad \therefore \quad 19 \bmod 7 = 5, \quad 18 \bmod 4 = 2, \quad 22 \bmod 5 = 2$$

- R is called the element of module P, and the element of P are $0, 1, 2, 3, 4$.
- If P is a prime number then all the 4 operations $(+, -, \times, \div)$ of the elements of P are possible and the results after these operators are also be the elements of the same module P.

Let $P = 7$ and elements are $0, 1, 2, 3, 4, 5, 6$.

Now,
$$3+4 = 7 = 7 \bmod 7 = 0$$
$$3-4 = 10-4 = 6$$
$$3\times 4 = 12 \bmod 7 = 5$$
$$3\div 4 = (3\times 7+3)\div 4 = 24\div 4 = 6$$

Galoi's field

When any element of a prime module (p) in multiplied in terms by all its non-zero elements each time a different product is obtained. This property does not held when pp is not a prime member. All division are not possible then so when all divisions $(+,-,\times,\div)$ are possible, the elements are said to from a finite field.

Example

If $P = 7$, the elements are $0,1,2,3,4,5,6$.

$$3\times 1 = 3 \quad ; \quad 3\times 2 = 6$$
$$3\times 3 = 9 \bmod 7 = 2 \quad ; \quad 3\times 4 = 12 \bmod 7 = 5$$
$$3\times 5 = 15 \bmod 7 = 1 \quad ; \quad 3\times 6 = 18 \bmod 7 = 4$$

Primitive root

There is at least one element in every field $GF(p)$, different power of which give the different non-zero elements of the field, such an element is called primitive root of the field.

Example

Let If $P=7$, the elements are $GF(p)\rightarrow 0,1,2,3,4,5,6.$

Now, $3^0=1$; $3^1=3$

$3^2=9 \mod 7=2$; $3^3=27 \mod 7=6$

$3^4=81 \mod 7=4$; $3^5=243 \mod 7=5$

So 3 is the primitive root of $GF(7)$ also, 2 and 6 are the primitive root of $GF(11)$

Application of Galois field

Some applications of Galois field in the experimental design are given below

- $BIBD$ can be constructed to using primitive root.
- $PBIBD$ is constructed.
- Construction of IBD from LSD using minimum function.
- Construction of orthogonal LSD using minimum function.
- An infinite set of numbers can be reduced.

Example: Construct a $BIBD$ having the parameters $v=9, b=12, r=4, k=3\, and\ \lambda=1$.

Solution

Here $k=3$, so there are $3-1=2$ orthogonal Latin square which are

$OLS-1$			$OLS-2$		
A	B	C	α	β	γ
B	C	A	γ	α	β
C	A	B	β	γ	α

Let the treatments are $1, 2, 3, 4, 5, 6, 7, 8, 9$. Now 9 treatments are arranged as 3×3 square

Blocks obtained from rows

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix} \quad \cdots \quad \cdots \quad \cdots \quad (A)$$

Blocks obtained from column

$$\begin{matrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{matrix}$$

Now we superimpose $OLS-1$ on (A) and $OLS-2$ on (A) we get,

Blocks obtained from Latin square

$$\begin{matrix} 1 & 6 & 8 \\ 2 & 4 & 9 \\ 3 & 5 & 7 \end{matrix}$$

Blocks obtained from orthogonal Latin square

$$\begin{matrix} 1 & 5 & 9 \\ 2 & 6 & 7 \\ 3 & 4 & 8 \end{matrix}$$

Thus the $BIND$ is

Blocks	*Contents*		
1	1	2	3
2	4	5	6
3	7	8	9
4	1	4	7
5	2	5	8
6	3	6	9
7	1	6	8
8	2	4	9
9	3	5	7
10	1	5	9
11	2	6	7
12	3	4	8

Example: Construct a BIBD having parameters $b = v = 13$, $r = k = 4$ and $\lambda = 1$.

Solution

Taking the elements of $\mathrm{mod}\, k = 13$ as the treatments. So the treatments are $0, 1, 3, 9$ from an initial block and these are shown below:

0	1	3	9	
	1	3	9	$1-0=1,\ 3-0=3,\ 9-0=9$
	12	10	4	$0-1=12,\ 0-3=10,\ 0-9=4$
		2	8	$3-1=2,\ 9-1=8$
		11	5	$1-3=11,\ 1-9=5$
			6	$9-3=6$
			7	$3-9=7$

It is see that among the above 12 differences each of the non-zero elements of $\mathrm{mod}\ 13$ occurs once. Hence by developing this block $\mathrm{mod}\ 13$ a symmetrical BIBD with parameters $b = v = 13,\ r = k = 4,\ \lambda = 1$ is obtained.

The actual design is shown below;

Blocks	Contents			
1	0	1	3	9
2	1	2	4	10
3	2	3	5	11
4	3	4	6	12
5	4	5	7	0
6	5	6	8	1
7	6	7	9	2
8	7	8	10	3
9	8	9	11	4
10	9	10	12	5
11	10	11	0	6
12	11	12	1	7
13	12	0	2	8

Construction of IBD using Primitive root

Construction of IBD of the series

$$b = v = 4\lambda + 3, \quad r = k = 2\lambda + 1, \quad \lambda$$

If $4\lambda + 3$ is a prime or a prime power, then the initial block formed of the even powers of the primitive root x of $GF(4\lambda + 3)$ gives the given series of BIBD.

The odd powers of the primitive roots also form another initial block which gives the same design.

Then the initial block is developed by adding in turn each of the different non-zero elements of the field toe the elements of the initial block.

Example: Construct a BIBD having parameters $b = v = 11, \quad r = k = 5, \quad \lambda = 2$.

Solution

The design $b = v = 11, \quad r = k = 5, \quad \lambda = 2$ is obtained by developing the initial block formed of the even powers of 2 which is the primitive root of $GF(11)$.

$$2^0 = 1, \quad 2^1 = 2, \quad 2^2 = 4, \quad 2^3 = 8, \quad 2^4 = 16 \bmod 11 = 5,$$
$$2^5 = 32 \bmod 11 = 10, \quad 2^6 = 64 \bmod 11 = 9, \quad 2^7 = 128 \bmod 11 = 7,$$
$$2^8 = 256 \bmod 11 = 3, \quad 2^9 = 512 \bmod 11 = 6, \quad 2^{10} = 102 \bmod 11 = 1.$$

If we consider the even powers then the initial block is

$$2^0, \quad 2^2, \quad 2^4, \quad 2^6, \quad 2^8$$
$$1, \quad 4, \quad 5, \quad 9, \quad 3$$

The actual design is show below:

Blocks	Contents				
1	1	4	5	9	3
2	2	5	6	10	4
3	3	6	7	0	5
4	4	7	8	1	6
5	5	8	9	2	7
6	6	9	10	3	8
7	7	10	0	4	9
8	8	0	1	5	10
9	9	1	2	6	0
10	10	2	3	7	1
11	0	3	4	8	2