Covariance Analysis

Main variable or Response variable:

The variable representing yield is called main variable or response variable. It is denoted by y.

Concomitant variables:

The additional variables representing heterogeneity of experimental units are called concomitant variables, auxiliary variables, supplementary variables or covariate denoted by $x_1, x_2, x_3, \dots, x_p$.

Layout plan:

Placement of the treatment to the experimental unit according to the principle of an experimental design is called lay-out plan. Let y_{ij} is the j^{th} observation receiving the i^{th} treatment, then the layout plan is

Treatment (t)			
t_1	t_2	$\dots$	t_p
y_{11}	y_{21}	$\dots$	y_{p1}
y_{12}	y_{22}	$\dots$	y_{p2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
y_{1q}	y_{2q}	$\dots$	y_{pq}

Analysis of covariance:

Analysis of covariance is the process of analysis of variance on the observation of response variable y after adjusting for the effects of uncontrolled concomitant variables.

In other words, analysis of covariance is the method testing for differences among the treatment means after adjusting the yields for the effects of one or more covariates. As a matter of fact, analysis of covariance is the combination of two components variance analysis and regression analysis.

Examples of analysis of covariance:

1. In agricultural experiments with fertilizer treatments the number of plants per plot prior to the application of fertilizer will also contribute to the variation in the observed yields. Here yield of a crop is main variable, and the number of plants is covariate.
2. In an experiment involving various teaching methods, the results of students can be adjusted for I.Q. before the experiment starts.
3. In animal feeding experiments the initial weights of animals under investigation can be used as covariate.
4. In an industrial experiment, by product may be found along with main product.
5. In petrochemical experiments, diesel is the main product while nylon, polyester etc. are by-products.

Uses of analysis of covariance:

The important uses of analysis of covariance are enumerated below-

1. Analysis of covariance (ANCOVA) can be applied to partition the total co-variation present in observed data into component parts.
2. It is used to explain the data and the model with treatment effect.

3. It helps to improve the efficiency of the experiment by controlling the experimental error.
4. It is used to estimate missing observations and to analyze the data with missing observations.
5. It helps to estimate treatment effect by eliminating the effect of concomitant variable.
6. It is used to partitioning of total co variation.
7. It is also used to determine the nature of treatment for valid interpretation of data.
8. It is applied to increased precision and error control of an experiment.
9. It is used to determine the adjusted treatment means.
10. It is used to test the homogeneity of regression equations.

Advantages of ANCOVA:

- ANCOVA adjusts for the effect of uncontrolled nuisance variables.
- ANCOVA controls experimental errors and increases precision.
- ANCOVA can be developed for non-orthogonal data.

Disadvantages of ANCOVA:

- If there is more than one external source of variation i.e. more than one concomitant variable, then the analysis becomes complicated.
- There may be certain types of observation which do not satisfy all the assumptions of ANCOVA.

Difference between Analysis of Covariance (ANCOVA) and Analysis of Variance (ANOVA):

1. ANOVA is the systematic procedure of partitioning the total variation present in a set of observations associated with the nature of classification of data. ANCOVA is the method of adjusting for the effects of an uncontrolled concomitant variable.
2. ANOVA is a technique for separating sample variance for a group of samples into partitions which are attributed to different sources, whereas ANCOVA is the combination of ANOVA and regression analysis.
3. ANOVA is used for testing the equality of several means, while ANCOVA is used for improving the precision of an experiment.
4. ANOVA deals with univariate data, whereas ANCOVA deals with bivariate data.

Analysis of Covariance in One-way Classification with Single Covariate or Completely Randomized Design with One Concomitant Variable:

The linear model for analysis of covariance in a completely randomized design (CRD) with a single covariate is given by

$$y_{ij} = \mu + \alpha_i + \beta(x_{ij} - \bar{x}_{..}) + e_{ij} \qquad ; \ i = 1\,(1)\,p \ \ , \ \ j = 1\,(1)q$$

where, y_{ij} is the j^{th} observation receiving the i^{th} treatment of the response variable

μ is the general mean effect

α_i is the i^{th} treatment effect

β is the regression coefficient of y on x

x_{ij} is the concomitant variable of the j^{th} observation receiving the i^{th} treatment related to response variable

e_{ij} is the random error component.

Assumption:

1. Response variable y is a random variable such that y values are normally distributed with common variance σ^2.
2. Covariate x is fixed whose values are measured without error.
3. The concomitant variable x is not affected by the treatments so that the variation of x values is not due to the treatments.
4. Regression of y on x is linear, such that $E(y|X) = \alpha + \beta(x_{ij} - \bar{x}_{..})$
5. Regression lines for k treatments are parallel straight lines having common or identical slopes or regression coefficients β for various treatments and $\beta \neq 0(\beta_i = \beta)$.
6. Regression effect is independent of treatment.
7. $\mu, \alpha_i and \beta$ are unknown parameters
8. $e_{ij} \sim NID(0, \sigma^2)$

Parameter Estimation:

By using the least square method we obtain the estimates of the unknown parameters.

The model is $\quad y_{ij} = \mu + \alpha_i + \beta$

$$(x_{ij} - \bar{x}_{..}) + e_{ij} \qquad ; \ i = 1\,(1)\,p \ \ , \ \ j = 1\,(1)q$$

$$\Phi = \sum_i^p \sum_j^q e_{ij}^2 = \sum_i^p \sum_j^q [y_{ij} - \mu - \alpha_i - \beta(x_{ij} - \bar{x}_{..})]^2 \ldots\ldots\ldots(1)$$

From equation (1) we obtain the following normal equations

$$y_{..} = pq\hat{\mu} + \hat{\beta}\sum_i \sum_j (x_{ij} - \bar{x}_{..}) \ldots\ldots\ldots(2)$$

$$y_{i.} = q\hat{\mu} + q\hat{\alpha}_i + \beta(x_{i.} - q\bar{x}_{..}) \ldots\ldots\ldots(3)$$

$$\therefore \sum_i \sum_j y_{ij}(x_{ij} - \bar{x}_{..}) = \hat{\mu}\sum_i \sum_j (x_{ij} - \bar{x}_{..}) + \sum_i \sum_j \hat{\alpha}_i(x_{ij} - \bar{x}_{..}) + \hat{\beta}\sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$= \sum_i \hat{\alpha}_i(x_{i.} - q\bar{x}_{..}) + \hat{\beta}\sum_i \sum_j (x_{ij} - \bar{x}_{..})^2 \ldots\ldots\ldots(4)$$

From the first equation, we obtain

$$\hat{\mu} = \bar{y}_{..} \qquad \left[\because \sum_i \sum_j (x_{ij} - \bar{x}_{..}) = 0\right]$$

From equation (3);

$$q\hat{\alpha}_i = y_{i.} - q\hat{\mu} - \hat{\beta} q(\bar{x}_{i.} - \bar{x}_{..})$$

$$\therefore \hat{\alpha}_i = \bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$$

From equation (4);

$$\sum_i \sum_j y_{ij}(x_{ij} - \bar{x}_{..}) = \sum_i \hat{\alpha}_i (x_{i.} - q\bar{x}_{..}) + \hat{\beta} \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2 \ldots\ldots\ldots (4)$$

$$\sum_i \sum_j (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..}) = \sum_i [\bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})](x_{i.} - q\bar{x}_{..}) + \hat{\beta} \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$= q\sum_i (\bar{y}_{i.} - \bar{y}_{..})(\bar{x}_{i.} - \bar{x}_{..}) - \hat{\beta} q \sum_i (\bar{x}_{i.} - \bar{x}_{..})^2 + \hat{\beta} \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

Now, let us define,

$$G_{yy} = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2$$

$$G_{xx} = \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$G_{xy} = \sum_i \sum_j (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..})$$

$$T_{yy} = q\sum_i (\bar{y}_{i.} - \bar{y}_{..})^2$$

$$T_{xx} = q\sum_i (\bar{x}_{i.} - \bar{x}_{..})^2$$

$$T_{xy} = q\sum_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..})$$

$$E_{xx} = G_{xx} - T_{xx}$$

$$E_{yy} = G_{yy} - T_{yy}$$

$$E_{xy} = G_{xy} - T_{xy}$$

$$\therefore G_{xy} = T_{xy} - \hat{\beta} T_{xx} + \hat{\beta} G_{xx}$$

$$\Rightarrow G_{xy} - T_{xy} = \hat{\beta}(G_{xx} - T_{xx})$$

$$\Rightarrow E_{xy} = \hat{\beta} E_{xx}$$

$$\therefore \hat{\beta} = \frac{E_{xy}}{E_{xx}}$$

How would you test the hypothesis that all the treatment effect are insignificant after eliminating the effect of covariate?

The model is $y_{ij} = \mu + \alpha_i + \beta(x_{ij} - \bar{x}_{..}) + e_{ij}$; $i = 1\,(1)\,p$, $j = 1\,(1)q$

$$\hat{\mu} = \bar{y}_{..};\ \hat{\alpha}_i = \bar{y}_{i.} - \bar{y}_{..} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..}) \text{ and } \hat{\beta} = \frac{E_{xy}}{E_{xx}}$$

Over the null hypothesis $H_0: \alpha_i = 0$

Now we must calculate different sum of squares, such as

$$SS(due\ to\ estimate) = \hat{\mu} y_{..} + \sum_i \hat{\alpha}_i y_{i.} + \sum_i \sum_j \hat{\beta}(x_{ij} - \bar{x}_{..}) y_{ij}$$

$$= \bar{y}_{..} y_{..} + \sum_i [(\bar{y}_{i.} - \bar{y}_{..}) - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})] y_{i.} + \hat{\beta} \sum_i \sum_j (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..})$$

$$= pq\bar{y}_{..}^2 + q\sum_i [(\bar{y}_{i.} - \bar{y}_{..}) - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})](\bar{y}_{i.} - \bar{y}_{..}) + \hat{\beta} G_{xy}$$

$$= pq\bar{y}_{..}^2 + q\sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 - \hat{\beta} q \sum_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..}) + \hat{\beta} G_{xy}$$

$$= pq\bar{y}_{..}^2 + T_{yy} - \hat{\beta} T_{xy} + \hat{\beta} G_{xy}$$

$$= pq\bar{y}_{..}^2 + T_{yy} + \hat{\beta}(G_{xy} - T_{xy}) \qquad [\because G_{xy} - T_{xy} = E_{xy}]$$

$$= pq\bar{y}_{..}^2 + T_{yy} + \hat{\beta} E_{xy} \ldots\ldots\ldots (1)$$

The degrees of freedom of (1) is

$$\text{number of parameters - } 1 = 1 + p + 1 - 1 = p + 1$$

$$Again, SS(due\ to\ error) = RawS.S - SS(due\ to\ estimate)$$

$$= \sum_i \sum_j y_{ij}^2 - pq\bar{y}_{..}^2 - T_{yy} - \hat{\beta}E_{xy}$$

$$= \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2 - T_{yy} - \hat{\beta}E_{xy} = G_{yy} - T_{yy} - \hat{\beta}E_{xy}$$

$$= E_{yy} - \hat{\beta}E_{xy} \ldots\ldots\ldots (2)$$

The degrees of freedom (2) is

$$= pq - (p + 1) = pq - p - 1$$

Let us now assume that our null hypothesis is true, then the model is

$$y_{ij} = \mu + \beta(x_{ij} - \bar{x}_{..}) + e_{ij} \qquad \ldots\ldots\ldots (A)$$

Now, we estimate the parameters μ and β by using OLS

From equation (A) we get the normal equations as

$$y_{..} = pq\hat{\mu} + \hat{\hat{\beta}} \sum\sum (x_{ij} - \bar{x}_{..})$$

$$\Rightarrow y_{..} = pq\hat{\mu} \because \sum\sum (x_{ij} - \bar{x}_{..}) = 0 \ldots\ldots\ldots (*)$$

$$\therefore \hat{\mu} = \bar{y}_{..}$$

And

$$\textstyle\sum_i \sum_j y_{ij}(x_{ij} - \bar{x}_{..}) = \mu \sum\sum (x_{ij} - \bar{x}_{..}) + \hat{\hat{\beta}} \sum\sum (x_{ij} - \bar{x}_{..})^2$$

$$\Rightarrow \sum_i \sum_j (y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..}) = \hat{\hat{\beta}} \sum\sum (x_{ij} - \bar{x}_{..})^2 \ldots\ldots\ldots (**)$$

$$\Rightarrow G_{xy} = \hat{\hat{\beta}} G_{xx}$$

$$\therefore \hat{\hat{\beta}} = \frac{G_{xy}}{G_{xx}}$$

$$SS(\text{due to estimates under } H_0\colon \alpha_i = 0 \text{ is true}) = \hat{\mu} y_{..} + \hat{\hat{\beta}} \sum\sum y_{ij}(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + \hat{\hat{\beta}} \sum\sum (y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + \hat{\hat{\beta}} G_{xy} \ldots\ldots\ldots (***)$$

The degree of freedom of equation $(***)$ is 2.

$$SS(\text{due to error under } H_0\colon \alpha_i = 0 \text{ is true}) = \text{Raw}S.S - SS(\text{due to estimate under } H_0\colon \alpha_i = 0 \text{ is true})$$

$$= \sum\sum y_{ij}^2 - pq\bar{y}_{..}^2 - \hat{\hat{\beta}} G_{xy}$$

$$= \sum\sum (y_{ij} - \bar{y}_{..})^2 - \hat{\hat{\beta}} G_{xy}$$

$$= G_{yy} - \hat{\hat{\beta}} G_{xy} \ldots\ldots\ldots (****)$$

The degrees of freedom of $(****)$ is

$$d.f \text{ raw } SS - d.f \text{ of } SS(\text{due to estimates under } H_0\colon \alpha_i = 0 \text{ is true}) = pq - 2$$

Now, we have,

Adjusted treatment $SS(\hat{\alpha}_i)$

$$= S.S(\text{due to error under } H_0\colon \alpha_i = 0 \text{ is true}) - S.S(\text{due to error under full model})$$

$$= G_{yy} - \hat{\hat{\beta}} G_{xy} - E_{yy} + \hat{\beta}E_{xy} = T_{yy} - \hat{\hat{\beta}} G_{xy} + \hat{\beta}E_{xy} \quad [\because G_{yy} - E_{yy} = T_{yy}]$$

Its degrees of freedom is

$$df\ of\ S.S(\text{due to error under } H_0: \alpha_i = 0 \text{ is true}) - df\ of\ S.S(\text{due to error under full model})$$
$$= (pq-2) - (pq-p-1)$$
$$= p-1$$

To test the hypothesis $H_0: \alpha_i = 0$ we have to use the following test statistic

$$F = \frac{\left(T_{yy}+\hat{\beta}E_{xy}-\hat{\hat{\beta}}G_{xy}\right)/_{(p-1)}}{\left(E_{yy}-\hat{\beta}E_{xy}\right)/_{(pq-p-1)}} = \frac{\text{Adjusted } MS \text{ of treatment}}{MSE}$$

Decision Rule:

If $F_{cal} < F_{\alpha\%,(p-1),(pq-p-1)}$ we accept the null hypothesis, that is there is no significance treatment effect.

ANCOVA Table:

S.V	$SS(x)$	$SP(xy)$	$SS(y)$	Regression coefficient	Adjusted $S.S(\hat{\alpha}_i)$
Treat	T_{xx}	T_{xy}	T_{yy}	$\hat{\beta} = \frac{E_{xy}}{E_{xx}}$	$T_{yy} + \hat{\beta}E_{xy} - \hat{\hat{\beta}}G_{xy}$
Error	E_{xx}	E_{xy}	E_{yy}		
Total	G_{xx}	G_{xy}	G_{yy}		

Test of hypothesis H_0: ⥂⥄⥂ $\beta = 0$:

To test the hypothesis H_0: ⥂⥄⥂ $\beta = 0$, let us assume that $\beta = 0$ is true, then the model becomes $y_{ij} = \mu + \alpha_i + e_{ij}$ and we get $\hat{\mu} = \bar{y}_{..}$ and $\hat{\alpha}_i = \bar{y}_{i.} - \bar{y}_{..}$

$$SS(\text{due to estimate if } H_0: \beta = 0 \text{ is true}) = \hat{\mu}y_{..} + \textstyle\sum_i \alpha_i y_i$$
$$= pq\bar{y}_{..}^2 + \sum_i (\bar{y}_{i.} - \bar{y}_{..})y_{i.}$$
$$= pq\bar{y}_{..}^2 + q\sum_i (\bar{y}_{i.} - \bar{y}_{..})\bar{y}_{i.}$$
$$= pq\bar{y}_{..}^2 + q\sum_i (\bar{y}_{i.} - \bar{y}_{..})(\bar{y}_{i.} - \bar{y}_{..})$$
$$= pq\bar{y}_{..}^2 + T_{yy} \ldots\ldots\ldots (1)$$

The degrees of freedom of (1) is

$$\text{Number of parameters- Number of restriction} = p + 1 - 1 = p$$

$$SS(\text{due to error if } H_0: \beta = 0 \text{ is true}) = \sum_i \sum_j y_{ij}^2 - pq\bar{y}_{..}^2 - T_{yy}$$
$$= \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2 - T_{yy} = G_{yy} - T_{yy} = E_{yy}$$

And degrees of freedom is $pq - p = p(q-1)$

$$SS(\text{due to regression}) = SS(\text{due to error if } H_0: \beta = 0 \text{ is true}) - SS(\text{due to error under full model})$$
$$= E_{yy} - E_{yy} + \hat{\beta}E_{xy}$$
$$= \hat{\beta}E_{xy}$$

And its degrees of freedom is $p(q-1) - (pq-p-1) = 1$

To test H_0: ⥂⥄⥂ $\beta = 0$ we use the following statistic

$$F = \frac{\hat{\beta} E_{xy} / 1}{(E_{yy} - \hat{\beta} E_{xy}) / (pq-p-1)}$$

Decision Rule:

If $F_{cal} < F_{tab} \sim F_{\alpha\%, 1, (pq-p-1)}$ then we can accept the null hypothesis, otherwise reject.

Again, to test $H_0: \rightleftarrows\leftrightarrows\rightleftarrows\ \beta = 0$ we use the following statistic

$$t = \frac{|\hat{\beta}|}{SE(\hat{\beta})} = \frac{|\hat{\beta}|}{\sqrt{\hat{\beta} E_{xy}}}$$

Decision Rule:

If $t_{cal} < t_{tab} \sim t_{\alpha/2\%, (pq-p-1)}$ then we can accept the null hypothesis, otherwise reject.

If $H_0: \leftrightarrows\rightleftarrows\rightleftarrows\ \beta = 0$ is accepted, then there is no justification for using concomitant variable.

If $H_0: \rightleftarrows\rightleftarrows\rightleftarrows\ \alpha_i = 0$ is rejected then how do you test for which level of treatment our null hypothesis is rejected?

If $H_0: \alpha_i = 0$ is rejected then we have to test the following hypothesis

$$H_0: \alpha_i = \alpha_{i'}; \qquad i \neq i' = 1(1)p$$

$$\Rightarrow H_0: \alpha_i - \alpha_{i'} = 0$$

Observed treatment mean $\bar{y}_{i.}$

Adjusted treatment mean $\bar{y}_{i.}$ can be obtained by $\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})$

$$var(adj\ \bar{y}_{i.}) = var\left[\bar{y}_{i.} - \hat{\beta}(\bar{x}_{i.} - \bar{x}_{..})\right]$$

$$= V(\bar{y}_{i.}) + (\bar{x}_{i.} - \bar{x}_{..})^2 V(\hat{\beta}) = \frac{\sigma^2}{q} + (\bar{x}_{i.} - \bar{x}_{..})^2 \frac{\sigma^2}{E_{xx}}$$

Now,

$$var\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right) = \frac{2\sigma^2}{q} + \left(\bar{x}_{i.} - \bar{x}_{i'.}\right)^2 \frac{\sigma^2}{E_{xx}}$$

$$var\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right) = MSE\left[\frac{2}{q} + \frac{\left(\bar{x}_{i.} - \bar{x}_{i'.}\right)^2}{E_{xx}}\right] \text{ if } \sigma^2 \text{is known}$$

To test $H_0: \alpha_i - \alpha_{i'} = 0 \quad ; \quad i \neq i' = 1(1)p$

We consider the test statistic as

$$t = \frac{\left|\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right)\right|}{SE\left|\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right)\right|} = \frac{\left|\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right)\right|}{\sqrt{MSE\left[\frac{2}{q} + \frac{\left(\bar{x}_{i.} - \bar{x}_{i'.}\right)^2}{E_{xx}}\right]}}$$

Decision Rule:

If $t_{cal} < t_{\alpha/2, (pq-p-1)}$ we cannot reject the null hypothesis, otherwise reject.

In agricultural research station an experiment is conducted to study the productivity of 2 varieties of potato using nitrogen fertilizer. The agricultural plots for cultivation are found homogeneous in respect of fertility. The potato varieties are randomly allocated to different plots. But the amount of fertilizer used (x kg/plot) in different plots are not same. The production of potato (y kg) in different plots along with amount of fertilizer used are given below:

Plot	Potato Varities			
	1		2	
	y	x	y	X
1	45	2	55	5
2	46	4	54	4
3	44	3	50	6

i. Analyze the data and group the varieties of potato which are similar in productivity.

ii. Do you think that the impact of concomitant variable is homogeneous for all varieties of potato.

Solution:

Plot	Potato Varities									
	1					2				
	y	x	y_sq	x_sq	xy	y	x	y_sq	x_sq	xy
1	45	2	2025	4	90	55	5	3025	25	275
2	46	4	2116	16	184	54	4	2916	16	216
3	44	3	1936	9	132	50	6	2500	36	300

$p = 2, q = 3, CT_{xx} = \frac{x_{..}^2}{pq}, CT_{yy} = \frac{y_{..}^2}{pq} \; CT_{yy} = \frac{x_{..}y_{..}}{pq}$

$$G_{xx} = \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2 = \sum_i \sum_j x_{ij}^2 - CT_{xx}$$

$$G_{yy} = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2 = \sum_i \sum_j y_{ij}^2 - CT_{yy}$$

$$G_{xy} = \sum_i \sum_j (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..}) = \sum_i \sum_j x_{ij}y_{ij} - CT_{xy}$$

$$T_{xx} = q \sum_i (\bar{x}_{i.} - \bar{x}_{..})^2 = \frac{\sum_i x_{i.}^2}{q} - CT_{xx}$$

$$T_{yy} = q \sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 = \frac{\sum_i y_{i.}^2}{q} - CT_{yy}$$

$$T_{xy} = q \sum_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..}) = \frac{\sum_i \bar{x}_{i.}\bar{y}_{i.}}{q} - CT_{xy}$$

For Full Model, $\hat{\beta} = \frac{E_{xy}}{E_{xx}}$, $\quad SSerror = (E_{yy} - \hat{\beta}E_{xy}); df = (pq - p - 1)$

Under $H_0: \alpha_i = 0$, $\quad \hat{\hat{\beta}} = \frac{G_{xy}}{G_{xx}}$

ANCOVA Table:

S.V	$SS(x)$	$SP(xy)$	$SS(y)$	Regression coefficient	Adjusted $S.S(\hat{\alpha}_i)$

Treat	T_{xx}	T_{xy}	T_{yy}	$\hat{\beta} = \frac{E_{xy}}{E_{xx}}$	$T_{yy} + \hat{\beta}E_{xy} - \hat{\hat{\beta}}G_{xy}$
Error	E_{xx}	E_{xy}	E_{yy}		
Total	G_{xx}	G_{xy}	G_{yy}		

Test of hypothesis $H_0: \alpha_i = 0$

$$F = \frac{\left(T_{yy} + \hat{\beta}E_{xy} - \hat{\hat{\beta}}G_{xy}\right)\Big/_{(p-1)}}{\left(E_{yy} - \hat{\beta}E_{xy}\right)\Big/_{(pq-p-1)}}$$

$$t = \frac{\left|\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right)\right|}{SE\left|\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right)\right|} = \frac{\left|\left(adj\bar{y}_{i.} - adj\bar{y}_{i'.}\right)\right|}{\sqrt{MSE\left[\frac{2}{q} + \frac{(\bar{x}_{i.} - \bar{x}_{i'.})^2}{E_{xx}}\right]}}$$

Test of hypothesis $H_0: \beta = 0$:

$$F = \frac{\hat{\beta}E_{xy}\Big/_{1}}{\left(E_{yy} - \hat{\beta}E_{xy}\right)\Big/_{(pq-p-1)}}$$

Analysis of Covariance in Randomized Block Design with one Concomitant Variable (RBD):

Randomized Block Design:

A randomized block design is a design in which the whole set of experimental units are arranged in several blocks which are internally homogeneous and treatments are randomly allocated to the experimental units within each block such that each treatment occurs once or some number of times in each block.

The linear model of analysis of covariance in RBD with concomitant is given by

$$y_{ij} = \mu + \alpha_i + \beta_j + \gamma\left(x_{ij} - \bar{x}_{..}\right) + e_{ij} \qquad ;\ i = 1\,(1)\,p\ \ ,\ \ j = 1\,(1)q$$

where, y_{ij} is the observation receiving the i^{th} block and j^{th} treatment

μ is general mean effect

α_i is the i^{th} block effect

β_j is the j^{th} treatment effect

γ is the regression coefficient of y on x

x_{ij} is the concomitant variable associated with y_{ij}

e_{ij} is the random error component.

Assumptions:

1. Response variable y is independently distributed with constant variance σ^2.
2. The concomitant variables $x's$ are fixed and measured without error.
3. Concomitant variable x is not influenced by the treatments.
4. Regression of y on x is linear.
5. Block effect, treatment effect and regression effects are additive.

6. $\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j = 0$.
7. Regression coefficient $\gamma \neq 0$.
8. $\mu, \alpha_i, \beta_j, and \gamma$ are unknown parameters.
9. $e_{ij} \sim NID(0, \sigma^2)$.

Parameters estimation:

By using OLS method we estimate the parameters of the model

$$y_{ij} = \mu + \alpha_i + \beta_j + \gamma(x_{ij} - \bar{x}_{..}) + e_{ij}$$

$$\therefore \Phi = \sum_i \sum_j e_{ij}^2 = \sum_i \sum_j [y_{ij} - \mu - \alpha_i - \beta_j - \gamma(x_{ij} - \bar{x}_{..})]^2 \ldots\ldots\ldots (1)$$

From equation (1) we obtain the following normal equations

$$y_{..} = pq\hat{\mu} + q\sum_i \hat{\alpha}_i + p\sum_j \hat{\beta}_j + \hat{\gamma}\sum_i \sum_j (x_{ij} - \bar{x}_{..}) \; [\because \sum_i \sum_j (x_{ij} - \bar{x}_{..}) = 0]$$

$$= pq\hat{\mu} + q\sum_i \hat{\alpha}_i + p\sum_j \hat{\beta}_j \ldots\ldots\ldots (2)$$

$$y_{i.} = q\hat{\mu} + q\hat{\alpha}_i + p\sum_j \hat{\beta}_j + \hat{\gamma}(x_{i.} - q\bar{x}_{..}) \ldots\ldots\ldots (3)$$

$$y_{.j} = p\hat{\mu} + q\sum_i \hat{\alpha}_i + p\hat{\beta}_j + \hat{\gamma}(x_{.j} - p\bar{x}_{..}) \ldots\ldots\ldots (4)$$

$$\sum_i \sum_j y_{ij}(x_{ij} - \bar{x}_{..}) = \hat{\mu}\sum_i \sum_j (x_{ij} - \bar{x}_{..}) + q\sum_i \hat{\alpha}_i(x_{ij} - \bar{x}_{..}) + p\sum_j \hat{\beta}_j(x_{ij} - \bar{x}_{..}) + \hat{\gamma}\sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$= q\sum_i \hat{\alpha}_i(x_{ij} - \bar{x}_{..}) + p\sum_j \hat{\beta}_j(x_{ij} - \bar{x}_{..}) + \hat{\gamma}\sum_i \sum_j (x_{ij} - \bar{x}_{..})^2 \ldots\ldots\ldots (5)$$

We see that the above equations are not independent. Hence we cannot get the unique solution of the parameters. To get unique solution of the parameters we have to put some restrictions. They are

$$\sum_{i=1}^{p} \alpha_i = \sum_{j=1}^{q} \beta_j$$

Let us define,

$$G_{xx} = \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$G_{yy} = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2$$

$$G_{xy} = \sum_i \sum_j (x_{ij} - \bar{x}_{..})(y_{ij} - \bar{y}_{..})$$

$$T_{xx} = p\sum_j (\bar{x}_{.j} - \bar{x}_{..})^2$$

$$T_{yy} = p\sum_j (y_{ij} - \bar{y}_{..})^2$$

$$T_{xy} = p\sum_j (\bar{x}_{.j} - \bar{x}_{..})(\bar{y}_{.j} - \bar{y}_{..})$$

$$B_{xx} = q\sum_i (\bar{x}_{i.} - \bar{x}_{..})^2$$

$$B_{yy} = q\sum_j (\bar{y}_{i.} - \bar{y}_{..})^2$$

$$B_{xy} = q\sum_i (\bar{x}_{i.} - \bar{x}_{..})(\bar{y}_{i.} - \bar{y}_{..})$$

$$E_{xx} = G_{xx} - B_{xx} - T_{xx}$$

$$E_{yy} = G_{yy} - B_{yy} - T_{yy}$$

$$E_{xy} = G_{xy} - B_{xy} - T_{xy}$$

Putting the restrictions using the above expression we get from the normal equations

$$\hat{\mu} = \bar{y}_{..}$$

$$\hat{\alpha}_i = \bar{y}_{i.} - \bar{y}_{..} - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})$$

$$\hat{\beta}_i = \bar{y}_{.j} - \bar{y}_{..} - \hat{\gamma}(\bar{x}_{.j} - \bar{x}_{..})$$

From (5) we obtain,

$$\sum_i \sum_j (y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..}) = q \sum_i [(\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})](\bar{x}_{i.} - \bar{x}_{..})$$

$$+p \sum_j [(\bar{y}_{.j} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{.j} - \bar{x}_{..})](\bar{x}_{.j} - \bar{x}_{..}) + \hat{\gamma} \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$= q \sum_i (\bar{y}_{i.} - \bar{y}_{..})(\bar{x}_{i.} - \bar{x}_{..}) - \hat{\gamma} q \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2$$

$$+p \sum_j (\bar{y}_{.j} - \bar{y}_{..})(\bar{x}_{.j} - \bar{x}_{..}) - \hat{\gamma} p \sum_i \sum_j (x_{ij} - \bar{x}_{..})^2 + \hat{\gamma} G_{xx}$$

$$\Rightarrow G_{xy} = B_{xy} - \hat{\gamma} B_{xx} + T_{xy} - \hat{\gamma} T_{xx} + \hat{\gamma} G_{xx}$$

$$\Rightarrow G_{xy} - B_{xy} - T_{xy} = \hat{\gamma}(G_{xx} - B_{xx} - T_{xx})$$

$$\Rightarrow E_{xy} = \hat{\gamma} E_{xx}$$

$$\therefore \hat{\gamma} = \frac{E_{xy}}{E_{xx}}$$

Now, we have to test whether there is any effect of treatment or not i.e. $H_0 : \beta_j = 0$.

First we conclude different sum of squares under the model

$$SS(\text{duetoestimate}) = \hat{\mu}y_{..} + \sum_i \hat{\alpha}_i y_{i.} + \sum_j \hat{\beta}_j y_{.j} + \hat{\gamma}\sum_i\sum_j y_{ij}(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + q\sum_i[(\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})]\bar{y}_{i.}$$

$$+p\sum_j[(\bar{y}_{.j} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{.j} - \bar{x}_{..})]\bar{y}_{.j} + \hat{\gamma}\sum_i\sum_j y_{ij}(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + q\sum_i[(\bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})](\bar{y}_{.j} - \bar{y}_{..})$$

$$+p\sum_j[(\bar{y}_{.j} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{.j} - \bar{x}_{..})](\bar{y}_{.j} - \bar{y}_{..}) + \hat{\gamma}\sum_i\sum_j(y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + B_{yy} - \hat{\gamma}B_{xy} + T_{yy} - \hat{\gamma}T_{xy} + \hat{\gamma}G_{xy}$$

$$= pq\bar{y}_{..}^2 + B_{yy} + T_{yy} + \hat{\gamma}(G_{xy} - B_{xy} - T_{xy})$$

$$= pq\bar{y}_{..}^2 + B_{yy} + T_{yy} + \hat{\gamma}E_{xy} = S_1 \ldots\ldots\ldots (*)$$

The degrees of freedom of $(*)$ is

Number of parameters - Number of restrictions $= (1 + p + q + 1)$-2 $= p + q$

$$SS(\text{due to error}) = \text{raw } SS - SS(\text{due to estimate})$$

$$= \sum_i\sum_j y_{ij}^2 - pq\bar{y}_{..}^2 - B_{yy} - T_{yy} - \hat{\gamma}E_{xy}$$

$$= \sum_i\sum_j(y_{ij} - \bar{y}_{..})^2 - B_{yy} - T_{yy} - \hat{\gamma}E_{xy}$$

$$= G_{yy} - B_{yy} - T_{yy} - \hat{\gamma}E_{xy}$$

$$= E_{yy} - \hat{\gamma}E_{xy} = S_2 \ldots\ldots\ldots (**)$$

The degrees of freedom of $(**)$ is $(pq - p - q)$

Our main object is to test whether there is any effect of treatment or not i.e. $H_0: \beta_j = 0$:

Under $H_0: \beta_j = 0$, the model becomes $y_{ij} = \mu + \alpha_i + \gamma(x_{ij} - \bar{x}_{..}) + e_{ij}$

Now, we apply OLS and find the normal equations as

$$y_{..} = pq\hat{\mu} + q\sum_i \hat{\alpha}_i + \hat{\hat{\gamma}}\sum_i\sum_j(x_{ij} - \bar{x}_{..}) = pq\hat{\mu} + q\sum_i \hat{\alpha}_i$$

$$y_{i.} = q\hat{\mu} + q\hat{\alpha}_i + \hat{\hat{\gamma}}(x_{i.} - q\bar{x}_{..})$$

$$\sum_i\sum_j y_{ij}(x_{ij} - \bar{x}_{..}) = \hat{\mu}\sum_i\sum_j(x_{ij} - \bar{x}_{..}) + q\sum_i \alpha_i(\bar{x}_{i.} - \bar{x}_{..}) + \hat{\hat{\gamma}}\sum_i\sum_j(x_{ij} - \bar{x}_{..})^2$$

From the above normal equations

$$\hat{\mu} = \bar{y}_{..}$$

$$\hat{\alpha}_i = (\bar{y}_{i.} - \bar{y}_{..}) - \hat{\hat{\gamma}}(\bar{x}_{i.} - \bar{x}_{..})$$

$$\sum_i\sum_j(y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..}) = q\sum_i[(\bar{y}_{i.} - \bar{y}_{..}) - \hat{\hat{\gamma}}(\bar{x}_{i.} - \bar{x}_{..})](\bar{x}_{i.} - \bar{x}_{..}) + \hat{\hat{\gamma}}\sum_i\sum_j(x_{ij} - \bar{x}_{..})^2$$

$$\Rightarrow G_{xy} = B_{xy} - \hat{\hat{\gamma}}B_{xx} + \hat{\hat{\gamma}}G_{xx}$$

$$\Rightarrow G_{xy} - B_{xy} = \hat{\hat{\gamma}}(G_{xx} - B_{xx})$$

$$\Rightarrow E_{xy} + T_{xy} = \hat{\hat{\gamma}}(E_{xx} + T_{xx})$$

$$\therefore \hat{\hat{\gamma}} = \frac{E_{xy} + T_{xy}}{E_{xx} + T_{xx}}$$

Now,

$$SS(\text{duetoestimate } |H_0\!:\beta_j = 0) = \hat{\mu} y_{..} + \textstyle\sum_i \hat{\alpha}_i y_{i.} + \hat{\hat{\gamma}} \sum_i \sum_j y_{ij}(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + q\sum_i [(\rightleftarrows\rightleftarrows \bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})]\bar{y}_{i.} + \hat{\gamma}\sum_i \sum_j (y_{ij} - \bar{y}_{..})(x_{ij} - \bar{x}_{..})$$

$$= pq\bar{y}_{..}^2 + q\sum_i [(\rightleftarrows\rightleftarrows \bar{y}_{i.} - \bar{y}_{..}) - \hat{\gamma}(\bar{x}_{i.} - \bar{x}_{..})](y_{ij} - \bar{y}_{..}) + \hat{\hat{\gamma}} G_{xy}$$

$$= pq\bar{y}_{..}^2 + B_{yy} - \hat{\hat{\gamma}} B_{xy} + \hat{\hat{\gamma}} G_{xy}$$

$$= pq\bar{y}_{..}^2 + B_{yy} - \hat{\hat{\gamma}}(G_{xy} - B_{xy})$$

$$= pq\bar{y}_{..}^2 + B_{yy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy}) = S_3 \ldots\ldots\ldots (***)$$

The degrees of freedom of $(***)$ is $(1 + p + 1) - 1 = p + 1$

$$SS(\text{duetoerror}|H_0) = \textstyle\sum_i \sum_j y_{ij}^2 - pq\bar{y}_{..}^2 - B_{yy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy})$$

$$= \sum_i \sum_j (y_{ij} - \bar{y}..)^2 - B_{yy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy})$$

$$= G_{yy} - B_{yy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy})$$

$$= E_{yy} + T_{yy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy}) = S_4 \ldots\ldots\ldots (****)$$

Its degrees of freedom is $(pq - p - 1)$

Now,

Adjusted treatment SS or $SS(\hat{\beta}_j) = SS(\text{duetoerror}|H_0) - SS(\text{duetoerrorunderfullmodel})$

$$= S_4 - S_2$$

$$= E_{yy} + T_{yy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy}) - E_{yy} + \hat{\gamma} E_{xy}$$

$$= T_{yy} + \hat{\gamma} E_{xy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy}) = S_5$$

Its degrees of freedom $= d.f \text{of} S_4 - d.f \text{of} S_2$

$$= (pq - p - 1) - (pq - p - q) = q - 1$$

To test the hypothesis $H_0\!: \beta_j = 0$ we consider the following test statistic

$$F = \frac{S_5/(q-1)}{S_2/(pq-p-q)} = \frac{[T_{yy} + \hat{\gamma} E_{xy} - \hat{\hat{\gamma}}(E_{xy} + T_{xy})]/(q-1)}{(E_{yy} - \hat{\gamma} E_{xy})/(pq-p-q)}$$

IF$F_{cal} < F_{\alpha\%,(q-1),(pq-p-q)}$, we accept the null hypothesis, otherwise we reject the null hypothesis.

ANCOVA Table:

S.V.	$d.f$	$SS(x)$	$SS(y)$	$SPxy$	Regression coefficient	Adjusted $S.S$	Under H_0 Adjusted $S.S$
Block	$P - 1$	B_{xx}	B_{yy}	B_{xy}			
Treat	$q - 1$	T_{xx}	T_{yy}	T_{xy}	$\gamma = \frac{E_{xy}}{E_{xx}}$	$E_{yy} - \hat{\gamma} E_{xy}$	E_{xx}
Error	$pq - p - q$	E_{xx}	E_{yy}	E_{xy}			
Treat + Error		$E_{xx} + T_{xx}$	$E_{yy} + T_{yy}$	$E_{xy} + T_{xy}$			

What is the justification for using one concomitant variable?

To test the justification for using one concomitant variable we have to test $H_0: \gamma = 0$. Under$H_0: \gamma = 0$, the model becomes $y_{ij} = \mu + \alpha_i + \beta_j + e_{ij}$

By using OLS we obtain the following normal equations

$$y_{..} = pq\hat{\mu} + q\sum_i \hat{\alpha}_i + p\sum_j \hat{\beta}_j$$

$$y_{i.} = q\hat{\mu} + q\hat{\alpha}_i + \sum_j \hat{\beta}_j$$

Since there two normal equations are not independent. So for getting unique solution we have to put some restrictions, such as $\sum_i \hat{\alpha}_i = 0$ and $\sum_j \hat{\beta}_j = 0$. Putting the restrictions we obtain

$$\hat{\mu} = \bar{y}..; \hat{\alpha}_i = \bar{y}_i. - \bar{y}.. and \hat{\beta}_j = \bar{y}._j - \bar{y}..$$

Now,

$$SS(\text{duetoestimatesunder}H_0: \gamma = 0) = \hat{\mu}y_{..} + \sum_i \hat{\alpha}_i y_{i.} + \sum_j \hat{\beta}_j y_{.j}$$

$$= pq\bar{y}_{..}^2 + \sum_i (\bar{y}_{i.} - \bar{y}_{..})y_{i.} + \sum_j (y_{.j} - \bar{y}_{..})y_{.j}$$

$$= pq\bar{y}_{..}^2 + q\sum_i (\bar{y}_{i.} - \bar{y}_{..})^2 + p\sum_j (y_{.j} - \bar{y}_{..})^2$$

$$= pq\bar{y}_{..}^2 + \sum_i (\bar{y}_{i.} - \bar{y}_{..})y_{i.} + \sum_j (y_{.j} - \bar{y}_{..})y_{.j}$$

$$= pq\bar{y}_{..}^2 + B_{yy} + T_{yy} = S_6$$

Its degrees of freedom is $(1 + p + q) - 2 = p + q - 1$

$$SS(\text{duetoerrorunder}H_0: \gamma = 0) = \sum_i \sum_j y_{ij}^2 - pq\bar{y}_{..}^2 - B_{yy} - T_{yy}$$

$$= \sum_i \sum_j (y_{ij} - \bar{y}..)^2 - B_{yy} - T_{yy} = G_{yy} - B_{yy} - T_{yy}$$

$$= E_{yy} = S_7$$

Its degrees of freedom is

$$pq - (p + q - 1) = pq - p - q + 1$$

$$= (p - 1)(q - 1)$$

$$\text{And}SS(duetoregression) = SS(\hat{\gamma}) = SS(duetoerror|H_0) - SS(duetoerrorunderfull)$$

$$= E_{yy} - E_{yy} + \hat{\gamma}E_{xy}$$

$$= \hat{\gamma}E_{xy} = S_8$$

Its degrees of freedom is $(pq - p - q + 1) - (pq - p - q) = 1$

To test the null hypothesis $H_0: \gamma = 0$ we consider the test statistic

$$F = \frac{RSS/_{d.f}}{SSE/_{d.f}} = \frac{\hat{\gamma}E_{xy}/_1}{(E_{yy} - \hat{\gamma}E_{xy})/_{(pq-p-q)}}$$

If $F_{cal} < F_{\alpha,1,(pq-p-q)}$, we accept the null hypothesis otherwise reject null hypothesis. If we reject the null hypothesis then we have need justification about the concomitant variable.

To test $H_0: \gamma = 0$ we consider the statistic

$$t = \frac{|\hat{\gamma}|}{\sqrt{\hat{\sigma}^2/_{E_{xx}}}} \text{where } \hat{\sigma}^2 = \frac{ESS}{pq-p-q}$$

If $t_{cal} < t\alpha/_{2,(pq-p-q)}$ we accept null hypothesis otherwise reject.

If treatment effect is insignificant, then need to test for which treatment the test is insignificant. Then we have to test the null hypothesis

$$H_0: \beta_j = \beta_{j'} j \neq j' = 1(1)q$$

$$\Rightarrow H_0: \beta_j - \beta_{j'} = 0 \text{which is contrast}$$

Adjusted treatment mean is calculated by $adj\bar{y}_{.j} = \bar{y}_{.j} - \hat{\gamma}(\bar{x}_{.j} - \bar{x}_{..})$

Now,

$$var(adj\bar{y}_{.j}) = var[\bar{y}_{.j} - \hat{\gamma}(\bar{x}_{.j} - \bar{x}_{..})]$$

$$= \frac{\sigma^2}{p} + (\bar{x}_{.j} - \bar{x}_{..})^2 \frac{\sigma^2}{E_{xx}}$$

$$\therefore var(adj\bar{y}_{.j} - adj\bar{y}_{.j'}) = \sigma^2 \left[\frac{2}{p} + \frac{(\bar{x}_{.j} - \bar{x}_{.j'})^2}{E_{xx}}\right]$$

To test $H_0: \beta_j = \beta_{j'}$ we use the statistic as

$$t = \frac{|adj\bar{y}_{.j} - adj\bar{y}_{.j'}|}{SE(adj\bar{y}_{.j} - adj\bar{y}_{.j'})} = \frac{|adj\bar{y}_{.j} - adj\bar{y}_{.j'}|}{\sqrt{MSE\left[\frac{2}{p} + \frac{(\bar{x}_{.j} - \bar{x}_{.j'})^2}{E_{xx}}\right]}}$$

If $t_{cal} < t\alpha/_{2,(pq-p-q)}$, we accept the null hypothesis otherwise reject.