02

The Measurement of Interest

Principal and Accumulated Value

Pri Accu

A common financial transition is one in which an amount of money is invested for a period of time. The amount of money initially invested is called the principal and the amount it has grown to after the time period is called the accumulated value at that time

Example: Suppose an amount of $1000\ Tk$ was invested for the time period 1 January 2004 to 1 January 2005. After the time period the amount becomes $1050\ Tk$. Here the initial amount of $1000\ Tk$ is the principal ~~accumulated~~ value at that time period.

Amount

Amount Function: If $t \geq 0$ is the length of time for which the principal has been invested, then the amount of money at that time will be denoted by $A(t)$, is called the amount function.

Mathematically, the amount function $A(t)$ is defined as:

$$A(t) = k.a(t) \qquad \text{where}, \quad k = A(0) = \text{principal } \textit{and}$$
$$a(t) = \text{accumulated value}$$

Accumulation Function

Acc fun

The simplest of all financial transactions is one in which an amount of money is invested for a period of time. The amount of money initially invested is called the principal and the amount it has grown to after the time period is called the accumulated value at that time.

Let t be the length of time for which the principle has been invested. Then the amount of money at that will be denoted by $A(t)$. This function is called amount function. Here we will consider $t \geq 0$ and assume that t is measured in years. The initial value $A(0)$ is just the principal itself.

Mathematically, we define the accumulation function from the amount function as

$$a(t) = \frac{A(t)}{A(0)}$$

Where , $a(0) = 1$ and $A(t)$ is just a constant multiple of $a(t)$ namely, $A(t) = k.a(t)$ where $k = A(0) = principle$.

Again we can say, any function $a(t)$ with $a(0) = 1$ could represent the way in which money accumulates with the passage of time is accumulation function. Constantly, however we would hope that $a(t)$ is increasing.

Properties Accu

Properties of Accumulation Function

The main properties of accumulation function are

i) $a(0) = 1$

ii) $a(t)$ is generally an increasing function.

iii) $a(t)$ is continuous if interest occures.

Type

Types of Accumulation Function

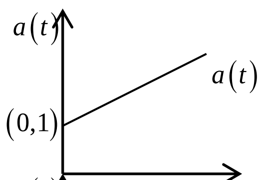
There are three types of accumulation functions. They are as follows:

1. Constant accumulation function.
2. Increasing accumulation function.
3. Step accumulation function.

constant

Constant Accumulation Function

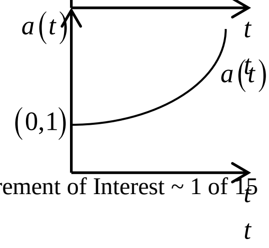
This is case, in which the amount of interest earned is constant over each year. The constant accumulation function can be represented as the following figure:



Increasing

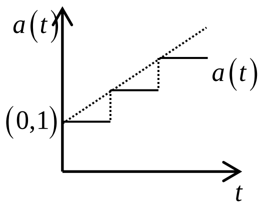
Increasing Accumulation Function

This is the case, in which the amount of interest earned is increasing as the year go on. The increasing accumulated as the following figure:



Step Accumulation Function

This is the case, in which interest is paid out at fixed periods of time, but no interest is paid if money is withdrawn these time periods. If the amount of interest paid is constant per time period, then the 'step' will all be the same height. The step accumulation can be represented as the following figure:



t

Interest

Interest

Interest may, be defined as the consider that a borrower of capital pays to a lender of capital for its use. Simple, the difference between the accumulated value and the principal is known as interest.

Mathematically, it can be expected as, $Interest = Accumulated\,Value - Principal$. Therefore, interest is nothing but the profit in an interest.

For example, suppose an amount of $1000\ Tk$ was invested for one year. After this time period the amount grown up to $1100\ Tk$. Then the interest is $\text{Interest} = 1100 - 1000 = 100$ Tk.

Standardized Measure for Interest (Effective Rate of Interest)

Standard Effective Rate

The effective rate of interest i is the amount of money that 1 invested at the beginning of a year will earn during the year, where interest is paid at the end of the year.

Or, the interest i earned on a principal of over a period of one year is called standardized measure for interest (effective rate of interest). That is

$$i = a(1) - a(0) \qquad \therefore \quad i = a(1) - 1 \qquad [\because \ a(0) = 1]$$

We can easily calculate i using the amount function $A(t)$ instead of $a(t)$.

We know that, $\quad A(t) = k.a(t)$

$$Thus, \qquad i = a(1) - 1 = \frac{a(1) - a(0)}{a(0)} = \frac{A(1) - A(0)}{A(0)}$$

Verbally, the effective rate of interest per year divided by the principal at the beginning of the year.

Let i_n be the effective rate of interest during the n^{th} year from the date of investment. Then we have,

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}$$

Example: Consider the function $a(t) = t^2 + t + 1$. Then

Ex 1

1) Verify that $a(0) = 1$.
2) Show that $a(t)$ is increasing for all $t \geq 0$.
3) Is $a(t)$ continuous?
4) Find the effective rate of interest, i for $a(t)$.
5) Find i_n.

Solution

1) We have, $\quad a(t) = t^2 + t + 1$

$$\therefore\ a(0) = 0^2 + 0 + 1$$
$$\therefore\ a(0) = 1 \quad (verified).$$

2) We have, $\quad a(t) = t^2 + t + 1$

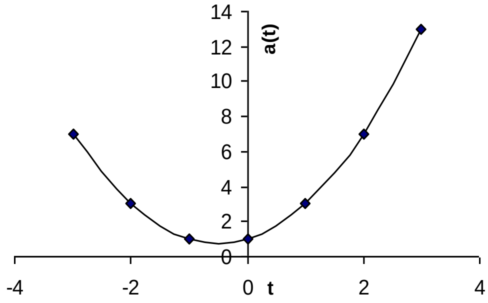
$$\therefore a'(t) = 2t + 1 > 0 \quad for\ all\ t \geq 0$$

So that, $a(t)$ is the increasing for all $t \geq 0$.

3) We have,

t	0	-1	1	-2	2	-3	3
$a(t)$	1	1	3	3	7	7	13

For the above results we draw the graph of $a(t)$ as:



We observe that the graph of $a(t)$ is a parabola, and hence $a(t)$ is continuous. Again, we know that all polynomial functions are continuous.

4) The effective rate of interest i or $a(t)$ is given by,

$$i = a(1) - 1$$

$$\text{But}, \quad \mathrm{a}(1) = 1+1+1 = 3$$

$$\therefore \quad i = 3 - 1 = 2.$$

So that the effective rate of interest i for $a(t)$ is 2.

5) We know that,

$$i_n = \frac{a(n) - a(n-1)}{a(n-1)} = \frac{n^2 + n + 1 - \left\{(n-1)^2 + (n-1) + 1\right\}}{(n-1)^2 + (n-1) + 1}$$

$$= \frac{n^2 + n + 1 - n^2 + 2n - 1 - n + 1 - 1}{n^2 - 2n + 1 + n - 1 + 1}$$

$$\therefore \quad i_n = \frac{2n}{n^2 - n + 1}$$

Types of Interest

There are two types of interest for the accumulation function $a(t)$. These are:

Type interest

1) Simple interest
2) Compound interest

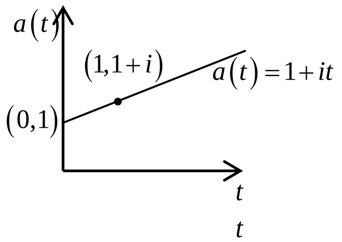
Simple Interest

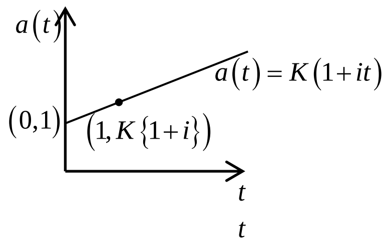
Simple interest is used occasionally, primarily between integer interest periods. In this situation, the amount of interest earned each year is constant. In other words, only the original principle earns interest from year to year and interest accumulated in any given year does not earn interest in the future years.

Simple

According to simple interest, if i is the effective rate of interest and k is the principle amount then the accumulated value after time t is

$$A(t) = k(1+it)$$





In case of simple interest, the graph becomes a straight line. The equation can be expressed as

$$a(t) = 1 + bt \quad \text{for some } b, \text{ since } a(0) = 1$$

The effective rate of interest is

$$i = a(1) - 1 = b$$

$$\therefore \quad a(t) = 1 + \left[a(1) - 1\right] t$$

$$\therefore \quad a(t) = 1 + it \quad ; \; t \geq 0$$

The formula $a(t) = 1 + it$, show the case where the principal is $A(0) = a(0) = 1$. More generally, if the amount of t will be

$$A(t) = k(1+it)$$

Simple interest can be estimate by the following methods:

i) Exact simple interest method

ii) Ordinary simple interest method or Banker's Rule.

Exact

Exact Simple Interest Method

In this method we use the exact number of days in the numerator and to use 365 days in the denominator. Interest computed on this basis is called exact simple interest. It is used in Canada.

Ordinary Simple Interest or Banker's Rule

The method in which we use the exact number of days in the numerator and to use 360 days in the denominator is called ordinary simple interest or Banker's Rule. It is used in USA.

Ex 2

Example: Assume Jack borrow 1000 tk from the bank or January 1, 1986 at a rate of 15% simple interest per year. How much does he owe on January 17, 1986?

Solution

We know that, the amount owing at time t is $A(t)=k(1+it)$

Two methods are common:

a) Exact simple interest method:

$$t=\frac{\text{Number of days}}{365}=\frac{16}{365}$$

$$\therefore \text{ We have, } \quad A(t)=1000\left(1+0.15\times\frac{16}{365}\right) \quad =1006.58$$

$\therefore$ Jack owes 1006.58 tk.

b) Ordinary simple interest:

$$t=\frac{\text{Number of days}}{360}=\frac{16}{360}$$

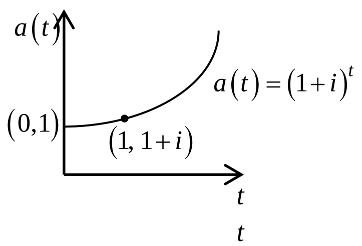
$$\therefore \text{ We have, } \quad A(t)=1000\left(1+0.15\times\frac{16}{360}\right) \quad =1006.67$$

$\therefore$ Jack owes 1006.67 tk.

Compound Interest

Continious

The most important case of accumulation function is the case of the compound interest. Intuitively speaking, this is the situation where money earns at a fixed effective rate. In this setting, the interest earned in one year, earns interest itself in the future years.



If i is the effective rate of interest then $a(1)=1+i$. So 1 becomes $1+i$ after the first year.

Let us partition $1+i$ in two parts as principal 1 and the interest i earned in first year. In the second year 1 will earn interest i, at the same time, interest i will earn interest $i.i=i^2$. So, total amount of interest earned in second year is

$$i+i^2=i(1+i)$$

Hence the total amount after 2 years is $a(2)=1+i+i(1+i)=(1+i)^2$.

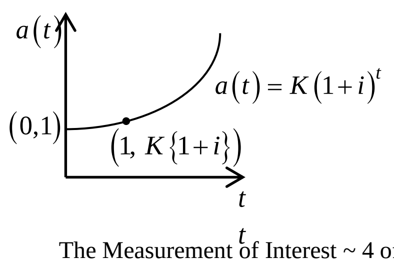
Proceeding in the similar way, we can obtain the general expression which is given as

$$a(t)=(1+i)^t \quad ; \quad t\geq 0.$$

The formula $a(t)=(1+i)^t$ shows the case where the principal is

$$A(0)=a(0)=1$$

More generally, if the principal at time $t=0$ is k, the amount at time t will be



$$A(t) = k(1+i)^t \quad ; \quad t \geq 0.$$

Example: If a man invests 100 Tk for two years at 5% simple interest, he will receive 5 Tk at the end of each of the two years. However, in reality, for the second year he has 105 Tk, which he could have invested had he started a new. Clearly, it would be to his advantage to invest the 105 Tk at 5%, since he would then receive 6.25 Tk in interest for the second year instead of 5 Tk.

Ex 3

Example: Jack borrows 1000 at a rate of 15% compound interest per year.

a) How much does he owe after 2 years?

b) How much does he owe after 57 days, assuming the compound interest between the integrals durations?

c) How much does he owe after 1 year and 57 days, under the same assumption as in part (b)?

d) How much does he owe after 1 year and 57 days, assuming the linear interpretation between the integrals durations?

e) In how many years will his principle has accumulated to 2000?

Solution

The general formula for the amount owing at time t in general is:

$$A(t) = k(1+i\ )^t$$

Where, $A(t) = Amount$, $k = \Pr inciple$, $i = Interest$, $t = Year\ or\ time$

a) So, Jack owes after 2 years,

$$A(t) = k(1+i)^t$$
$$= 1000(1+0.15)^2$$
$$\therefore\ A(2) = 1322.50 \text{ Tk}$$

b) The most suitable value for t is

$$t = \frac{\text{Number of days}}{\text{Number of days in a year}} = \frac{57}{365}$$

The accumulated value is

$$A\left(\frac{57}{365}\right) = 1000(1+0.15)^{\frac{57}{365}}$$
$$= 1022.07 \text{ Tk}$$

c) He owes after 1 year and 57 days

$$A\left(1\frac{57}{365}\right) = 1000(1+0.15)^{1\frac{57}{365}}$$
$$= 1175.38 \text{ Tk}$$

d) We most interpolate between $A(1)$ and $A(2)$.

We have,

$$A(1) = 1000(1+0.15) = 1150.00$$
$$\text{and} \quad A(2) = 1000(1+0.15)^2 = 1322.50$$

The difference between these values is

$$A(2) - A(1) = 1322.50 - 1150.00$$
$$= 172.50$$

The partition of this difference which will accumulate in 57 days, assuming simple interest is $\left(\frac{57}{365}\right) \times 172.50 = 26.94$

Thus the accumulated value after 1 year and 57 days is $1150.00 + 26.94 = 1176.94$

e) We find t such that

$$1000(1+0.15)^t = 2000$$

$$\Rightarrow \quad (1.15)^t = 2$$

$$\Rightarrow \quad t\log_{10}(1.15) = \log_{10} 2$$

$$\Rightarrow \quad t = \frac{\log_{10} 2}{\log_{10}(1.15)}$$

$$\therefore \quad t = 4.96 \approx 5 \text{ years}$$

Relation between Effective Rate of Interest and Simple Interest

We know that the effective rate of interest

$$i_n = \frac{a(n)-a(n-1)}{a(n-1)}$$

In case of simple interest we know that,

$$a(t) = 1+it$$

So, we have,

$$i_n = \frac{a(n)-a(n-1)}{a(n-1)} = \frac{1+in-[1+i(n-1)]}{1+i(n-1)}$$

$$= \frac{1+in-1-in+i}{1+i(n-1)}$$

$$\therefore \quad i_n = \frac{i}{1+i(n-1)}$$

Again, in case of compound interest we know that $a(t) = (1+it)^t$

So, we have,

$$i_n = \frac{a(n)-a(n-1)}{a(n-1)} = \frac{(1+i)^n-(1+i)^{n-1}}{(1+i)^{n-1}}$$

$$= \frac{(1+i)^{n-1}(1+i-1)}{(1+i)^{n-1}} = i$$

$$\therefore \quad i_n = i$$

i.e. in case of compound interest effective rate of interest and compound interest are same.

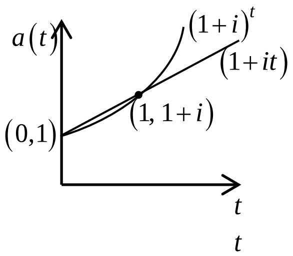
Comparison between Simple and Compound Interest

Simple Vs Compound

i) Simple and compound interest produce same return over a one year period.

ii) Compound interest is used almost exclusively for financial transactions covering, a period of one year or more and is often used for short term transactions as well.

Simple interest is occasionally used for short term transactions and as an approximation for compound interest over financial periods.



iii) The graph of compound interest is always concave up, but the graph of simple interest in straight line.

iv) Over a longer period of more than one year, compound interest produces a larger accumulated value than simple interest. Whereas, over a shorter period of less than one year, simple interest produces a larger accumulated value than compound interest i.e.

$$(1+it)^t > (1+it) \quad \text{for } t>1$$

$$(1+it)^t < (1+it) \quad \text{for } 0<t<1.$$

From the above relation we can conclude that the simple interest yields a higher return than the compound interest if $0<t<1$ and the compound interest yields a higher return than the simple interest if $t>1$. The first of these statements does not surprise us, since for $t > 1$, we have interest as well as principle earning interest in the $(1+i)^t$ case. The second statement reminds us that for the period of less than one year, simple interest is more beneficial to lend than the compound interest. If the periods of time equal to one-year than the simple interest is equal to the compound interest.

v) The formula for compound interest $a(t) = (1+it)^t$; $t \geq 0$. Whereas, the formula for simple interest is $a(t) = (1+it)$; $t \geq 0$

vi) Rate of compound interest and rate of effective interest are identical i.e. $i_n = i$. Whereas, rate of simple interest and rate of effective interest are not identical, i.e. $i_n = \dfrac{i}{1+i(n-1)}$

Which interest method should you prefer in Bangladesh in financial situation and why?

Interest best

As a loaner, I prefer simple interest for the long times period because for a long time in case of simple interest the accumulated value is lower than the accumulated value in case of compound interest. But for within one year I prefer the compound interest because hence the accumulated value in case of compound interest is lower than the accumulated value in case of simple interest. On the other hand, as a winner, I prefer the compound interest for the duration of long time period, because hence I get more profit than simple interest. But for the short time period i.e., if the duration is less than one year then I prefer the simple interest because hence I get more benefit than compound interest.

In respect of Bangladesh, we prefer simple interest. Because Bangladesh is a poor country, hence most of the people are owe and they does not pay their loan within one year.

Present Value

Present Value

The present value of t years in the past is the amount of money that will accumulate to the principal over t years. It is denoted by V . The term V is often called a discount factor.

Figure:

For example, 1 accumulate to $1+i$ over a single year. Let V be the amount of money at the present time, to accumulate to 1 over one year. So, V accumulated to $V(1+i)$.

Thus we have,

$$V(1+i) = 1$$
$$\therefore\ V = \frac{1}{1+i} = (1+i)^{-1}$$

These two accumulations are shown in the following figure

Figure:

We assume that we deal with compound interest, where $a(t) = (1+it)^t$. Thus the present value of 1, t years in the past will be

$$V^t = (1+i)^{-t}$$

We summarize this on the time diagram shown in below:

Figure:

The accumulation function for simple interest is $a(t) = 1+it$.

Hence present value of one unit t years in the past is given by V where,

$$V(1+it) = 1$$
$$\therefore\ V = (1+it)^{-1}$$

Figure:

Example: The Kelly family buys a new house for 93500 on May 1,1986. How much was this house worth in May1, 1982 if the real state prices have risen at a

Ex 4

1) Simple interest rate 8% per year during that period?

2) Compound interest rate 8% per year during that period?

Solution

1) Here the interest rate, $i = 8\% = 0.08$ and time $t = 4$ years.

Let the present value be x. We have to find the present value at time $t = -4$ of 93500 at time 0. According to the question with the simple interest, we have that

$$\begin{aligned} & x(1+it) = 93500 \\ \Rightarrow\ & x(1+0.08\times 4) = 93500 \\ \Rightarrow\ & 1.32x = 93500 \\ \Rightarrow\ & x = 70833.33 \end{aligned}$$

So, the house worth on May 1,1982 is 70833.33.

2) Here, we have that interest rate, $i = 8\% = 0.08$ and time, $t = 4$ years.

Let the present value be x. We have to find the present value at time $t = -4$ of 93500 at time 0. According to the question with the compound interest, we have that

$$\begin{aligned} & x(1+i)^t = 93500 \\ \Rightarrow\ & x(1+0.08)^4 = 93500 \\ \Rightarrow\ & x = \frac{93500}{(1.08)^4} \\ \Rightarrow\ & x = 68725.29 \end{aligned}$$

So, the house worth on May 1,1982 is 68725.29.

Discount

Discount

Paid at the beginning of the year on the balance at the end of the year is known as discount.

Discount focuses on the total at the end of the year. It is natural to define the effective rate of discount, d as

$$d = \frac{a(1)-1}{a(1)}$$

In other words, standardization is achieved by dividing by $a(1)$ instead of $a(0)$ in the denominator to define the effective rate of interest i.

More generally, the effective rate of discount in the n^{th} year is given by

$$d_n = \frac{a(n)-a(n-1)}{a(n)}$$

For example, if A borrows $100 for one year at an effective rate of discount of 6% from a bank, then the bank will collect its interest of 6% in advance and will give A only $94. At the end of the year A will repay $100.

Properties of Discount

Properties dis

i) A constant rate of simple discount implies an increasing effective rate of discount.

ii) Simple and compound discount produce the same result over a one year period.

iii) Over a longer period, simple discount produces a smaller present value than compound discount.

Uses of Discount

i) Simple discount is used only for short-term transactions.

ii) Discount is used in connection with present values (discount factor, discount function, discounting, and discounted value).

Use

Some Basic Relationship

If i is the effective rate of interest, d is the effective rate of discount V is the present value or discount factor. Then,

a) From the definition of d , we have,

Relation inter = dis

$$d = \frac{a(1)-1}{a(1)} = \frac{1+i-1}{1+i} = \frac{i}{1+i}$$

$$\therefore \quad d = \frac{i}{1+i}$$

Since $(1+i) > 1$, thus $i > d$.

b) Since $d = \frac{i}{1+i} = iv$

$$\therefore \; d = iv \qquad \text{where} \qquad v = \frac{1}{1+i}.$$

c) $$d = \frac{i}{1+i}$$

$$\Rightarrow \quad d + id = i$$

$$\therefore \quad i = \frac{d}{1-d}.$$

d)

$$d = \frac{i}{1+i}$$

$$= \frac{1+i}{1+i} - \frac{1}{1+i} = 1 - v$$

$$\therefore \quad d = 1 - v$$

$$\text{or}, \quad v = 1 - d.$$

Comparison between Interest and Discount

Interest Vs Dist

i) The effective rate of discount in the n^{th} year is given by

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{a(n) - a(n-1)}{a(n)}$$

$$d_n = \frac{I_n}{a(n)} \quad ; \quad n \geq 1.$$

On the other hand, the effective rate of interest in the n^{th} year is given by

$$i_n = \frac{A(n) - A(n-1)}{A(n-1)} = \frac{a(n) - a(n-1)}{a(n-1)}$$

$$i_n = \frac{I_n}{a(n-1)} \quad ; \quad n \geq 1.$$

ii) Discount is paid at the beginning of the year on the balance the end of the year.

Interest is paid at the end of the year on the balance at the beginning of the year.

iii) Two rates of discount and interest are said to be equivalent if a given principal invested for the same length of time.

Example: 1000 is to be accumulated by January 1, 1985, at a compound rate of discount of 9% per year.

a) Find the present value on January 1, 1982.

b) Find the value of i corresponding to d .

Ex 5

Solution

a) Here the rate of discount, $d = 0.09$ and time $t = 3$ years.

Let the present value be x . We have to find the present value at time $t = -3$ of 1000 at time 0 . According to the question with the compound interest, we have that,

$$x(1-d)^{-t} = 1000$$

$$\Rightarrow x(1-0.09)^{-3} = 1000$$

$$\Rightarrow x = \frac{1000}{(1-0.09)^4}$$

$$\Rightarrow x = 753.571$$

So, the present value on January 1, 1982 is 753.57.

b) Again, we know that,

$$i = \frac{d}{1-d} = \frac{0.09}{1-0.09} = 0.098901098 = 0.0989$$

So, the value of i corresponding to d is 0.0989.

Example: Jane deposits 1000 in a Bank account on August 1, 1986. If the rate of the compound interest is 7% per year, find the value of this deposit on August 1, 1984.

Ex 6

Solution

We think that the answer to this question should be zero, because the money has not been deposited yet! However, in a mathematical sense, we know that the money has value at all times, past or future.

Here the interest rate, $i = 7\% = 0.07$ and time, $t = 2$ years.

Let the present value be x. We have to find the present value at time $t = -2$ of 1000 at time 0. According to the question with the compound interest, we have that

$$x(1+i)^{t} = 1000$$

$$\Rightarrow x(1+0.07)^{2} = 1000$$

$$\Rightarrow x = \frac{1000}{(1.07)^2}$$

$$\therefore \quad x = 873.44$$

So, the value of this deposit on August 1, 1984 is 873.44.

Effective Rate of Discount

Effective

The effective rate of discount d is the ratio of the amount of interest (usually called the amount of discount or just discount) earned during the year to the amount inverted at the end of the year. Effective rate of discount can be calculated over any one year period. Let d_n be the effective rate of discount during the n^{th} year from the date of investment, then we have,

$$d_n = \frac{A(n) - A(n-1)}{A(n)} = \frac{I_n}{A(n)} \quad \text{for} \quad n \geq 1.$$

Nominal Rate of Interest

Nominal

Nominal rate of interest is denoted by $i^{(m)}$, convert able m times per year, which implies an effective rate of interest of $\frac{i^{(m)}}{m}$ per m^{th} of a year.

If i is the effective rate of interest per year, it follows that

$$1 + i = \left[1 + \frac{i^{(m)}}{m}\right]^m$$

$$\Rightarrow \quad 1 + \frac{i^{(m)}}{m} = (1+i)^{1/m}$$

$$\Rightarrow \quad i^{(m)} = m\left[(1+i)^{1/m} - 1\right]$$

Example: Consider a well known credit card which charges 18% per year convertible monthly. This means that the actual rate of interest is $\frac{0.18}{12}=0.015$ effective per month over the course of a year, 1 will accumulate to $(1+0.015)^{12}=1.1956$, so the effective rate of interest per year is actually 19.56%. Thus 18% is the nominal rate of interest.

Nominal Rate of Discount

The nominal rate of discount $d^{(m)}$, as meaning an effective rate of discount of $\frac{d^{(m)}}{m}$ per m^{th} of a year.

If d is the effective rate of discount if follows that

$$1-d=\left[1-\frac{d^{(m)}}{m}\right]^m$$

$$\Rightarrow \quad (1-d)^{1/m}=1-\frac{d^{(m)}}{m}$$

$$\Rightarrow \quad \frac{d^{(m)}}{m}=1-(1-d)^{1/m}$$

$$\therefore \quad d^{(m)}=m\left[1-(1-d)^{1/m}\right].$$

Relation between $i^{(m)}$ and $d^{(m)}$

$i^{(m)} / d^{(m)}$

We know that, $1-d=\left[1-\frac{d^{(m)}}{m}\right]^m$

Since $1-d=\frac{1}{1+i}$, we have,

$$\left[1+\frac{i^{(m)}}{m}\right]^m=1+i=\frac{1}{1-d}=\left[1-\frac{d^{(n)}}{n}\right]^{-n}$$ for all positive integers m and n.

$$\Rightarrow \quad \left[1+\frac{i^{(m)}}{m}\right]^m=\left[1-\frac{d^{(m)}}{m}\right]^{-m}$$

$$\Rightarrow \quad 1+\frac{i^{(m)}}{m}=\frac{1}{1-\frac{d^{(m)}}{m}}$$

$$\Rightarrow \quad 1-\frac{d^{(m)}}{m}=\frac{1}{1+\frac{i^{(m)}}{m}}$$

$$\Rightarrow \quad \frac{d^{(m)}}{m}=1-\frac{1}{1+\frac{i^{(m)}}{m}}$$

$$\Rightarrow \quad d^{(m)}=m\left[\frac{1+\frac{i^{(m)}}{m}-1}{1+\frac{i^{(m)}}{m}}\right]$$

$$\therefore \quad d^{(m)}=\frac{i^{(m)}}{1+\frac{i^{(m)}}{m}}$$

which is the relation between $d^{(m)}$ and $i^{(m)}$.

Example (Nominal Rate of Interest): A man borrows 1000 at an effective rate of interest 2% per month. How much does he owe after 3 years?

Solution

Since the effective rate of interest is given per month, three years is 36 interest periods. Thus the answer is

Ex 7

$$1000(1+0.02)^{36} = 2039.89 .$$

Example: Find the accumulated value of 1000 after three years at a rate of interest of 24% per year convertible monthly.

Solution

We know that,

$$1+i = \left[1+\frac{i^{(m)}}{m}\right]^m$$

$$\therefore \quad i = \left[1+\frac{i^{(m)}}{m}\right]^m - 1 \qquad\qquad \left| \; \frac{24}{12} = 2\% = 0.02\right.$$

$$= [1+0.02]^{12} - 1$$

$$\therefore \quad i = 0.26824$$

$$\therefore \quad \text{Accumulated value} = k(1+i)^t$$

$$= 1000(1+0.26824)^3$$

$$= 2039.89$$

Example: If $i^{(6)} = 0.15$, find the equivalent nominal rate of interest convertible semiannually.

Solution: We have, nominal rate of interest (semiannually) is $i^{(2)}$ and follows that $1+i = \left[1+\frac{i^2}{2}\right]^2$.

Since $i^{(6)} = 0.15$, we have, $1+i = \left[1+\frac{i^{(6)}}{6}\right]^6 = \left[1+\frac{0.15}{6}\right]^6$.

Thus we can write,

$$\left[1+\frac{i^{(2)}}{2}\right]^2 = \left[1+\frac{0.15}{6}\right]^6$$

$$\Rightarrow \quad 1+\frac{i^{(2)}}{2} = \left[1+\frac{0.15}{6}\right]^3$$

$$\Rightarrow \quad i^{(2)} = 2\left[(1.025)^3 - 1\right]$$

$$\therefore \quad i^{(2)} = 0.15378$$

Example: Find the nominal rate of discount convertible semiannually which is equivalent to a nominal rate of interest of 12% per year convertible monthly.

Solution

We know that, $\left[1-\frac{d^{(n)}}{n}\right]^n = \left[1+\frac{i^{(m)}}{m}\right]^{-m}$

Hence we have,

$$\left[1-\frac{d^{(2)}}{2}\right]^{2}=\left[1+\frac{i^{(12)}}{12}\right]^{-12}$$

$$\Rightarrow \quad 1-\frac{d^{(2)}}{2}=\left[1+\frac{0.12}{12}\right]^{-6}$$

$$\Rightarrow \quad \frac{d^{(2)}}{2}=1-\left[1+\frac{0.12}{12}\right]^{-6}$$

$$\therefore \; d^{(2)}=2\left[1-0.94204\right]$$

$$=0.11592$$

Force of Interest

It is important in many cases to be able to measure the intensity with which interest is operating at each moment of time i.e. over an infinitesimally small at individual moments of time is called the force of interest.

The force of interest at time t is denoted by δ_t and is defined as

$$\delta_t=\frac{D\left[A(t)\right]}{A(t)}=\frac{D\left[a(t)\right]}{a(t)}$$

Where, $D\left[A(t)\right]=$ Rate of change (or the slope of the $A(t)$ curve) at time t.

$a(t)=$ Accumulation function.

D stands for the derivative with respect to t.

Properties of the Force of Interest

i) It is a measure of interest at exact time t.

ii) It expresses this interest in the farm of an annual rate.

Derivation of δ_t

If $i^{(m)}$ denotes a nominal rate of interest convertible m tomes per year and i is the effective rate of interest, then

$$1+i=\left[1+\frac{i^{(m)}}{m}\right]^{m}$$

$$\Rightarrow \quad 1+\frac{i^{(m)}}{m}=(1+i)^{1/m}$$

$$\Rightarrow \quad i^{(m)}=m\left[(1+i)^{1/m}-1\right]$$

Taking limit on both sides we have,

$$\lim_{m\to\infty}=\lim_{m\to\infty} m\left[(1+i)^{1/m}-1\right]$$

$$=\lim_{m\to\infty}\frac{(1+i)^{1/m}-1}{1/m}$$

which is of the form $\frac{0}{0}$, we take derivatives top and bottom, cancel and obtain

$$\lim_{m\to\infty} i^{(m)}=\lim_{m\to\infty}\left[(1+i)^{1/m}\ln(1+i)\right]$$

$$=\ln(1+i) \qquad \text{since } \lim_{m\to\infty}(1+i)^{1/m}=1$$

This limit is called the force of interest and is denoted by δ. Thus we have,

$$\delta = \ln(1+i)$$
$$\therefore \; e^{\delta} = 1+i.$$

Remark

We observe that

$$D\left[(1+i)^t\right] = (1+i)^t \ln(1+i)$$

where D stands for the derivative with respect to t.

Theorem: Force of interest is equivalent to force of discount i.e. $\delta_t = \delta_t'$.

Proof

We know that,

$$\delta_t' = \frac{-D\left[a^{-1}(t)\right]}{a^{-1}(t)}$$
$$= \frac{a^{-2}(t)D\left[a(t)\right]}{a^{-1}(t)}$$
$$= \frac{a^{-2}(t).a(t).\delta_t}{a^{-1}(t)} \qquad \Bigg| \; \therefore \; \delta_t = \frac{D\left[a(t)\right]}{a(t)}$$
$$\therefore \; \delta_t' = \delta_t$$

(Proved)

Problem: Find δ_t in the case of simple interest.

Solution

We know that,

$$\delta_t = \frac{D\left[a(t)\right]}{a(t)}$$
$$= \frac{D[1+it]}{1+it}$$
$$= D\left[\ln(1+it)\right]$$
$$\therefore \; \delta_t = \frac{i}{1+it}$$

Problem: If δ_t is given find $a(t)$.

Solution

We know that, $\qquad \delta_r = \dfrac{D\left[a(r)\right]}{a(r)} = D\left[\ln a(r)\right]$

Integrating both sides from 0 to t, we have,

$$\int_0^t \delta_r dr = \int_0^t D\left[\ln(a(r))\right]dr$$
$$= \left[\ln a(r)\right]_0^t$$
$$= \ln a(t) - \ln a(0)$$
$$= \ln a(t)$$
$$\therefore \; a(t) = e^{\int_0^t \delta_r dr}$$

Theorem: Prove that if δ is a constant (i.e. independent of r), then $a(t) = (1+i)^t$ for some i.

Proof: We know that, $a(t) = e^{\int_0^t \delta_r dr}$

If $\delta_r = c$, then

$$a(t) = e^{\int_0^t c dr}$$

$$= e^{ct}$$

$$= \left(e^c\right)^t$$

$$\therefore \quad a(t) = \left(e^c - 1 + 1\right)^t$$

$$= (1+i)^t \qquad \text{where} \quad i = e^c - 1$$

Theorem: Problem: Prove that $\int_0^n A(t)\delta_t dt = A(n) - A(0)$ for any amount function $A(t)$.

Proof

$$\text{L.H.S} = \int_0^n A(t)\delta_t dt \qquad = \int_0^n A(t)\frac{D\left[A(t)\right]}{A(t)}dt$$

$$= \int_0^n D\left[A(t)\right]dt$$

$$= \left[A(t)\right]_0^n$$

$$= A(n) - A(0)$$

(Proved)

Example: Find $a(t)$ if $\delta_t = 0.04(1+t)^{-1}$.

Solution

We know that,

$$a(t) = e^{\int_0^n \delta_r dr}$$

$$= e^{\int_0^n 0.04(1+r)^{-1} dr}$$

$$= e^{0.04\int_0^n (1+r)^{-1} dr}$$

$$= e^{0.04\ln(1+t)}$$

$$\therefore \ a(t) = (1+t)^{0.04}.$$