

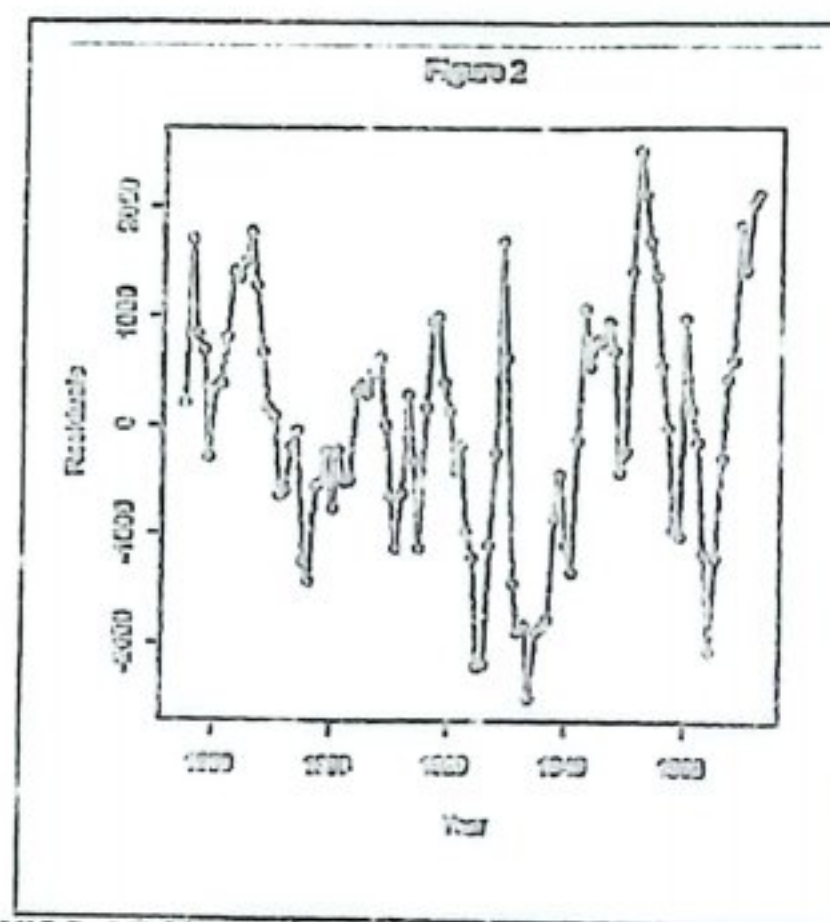
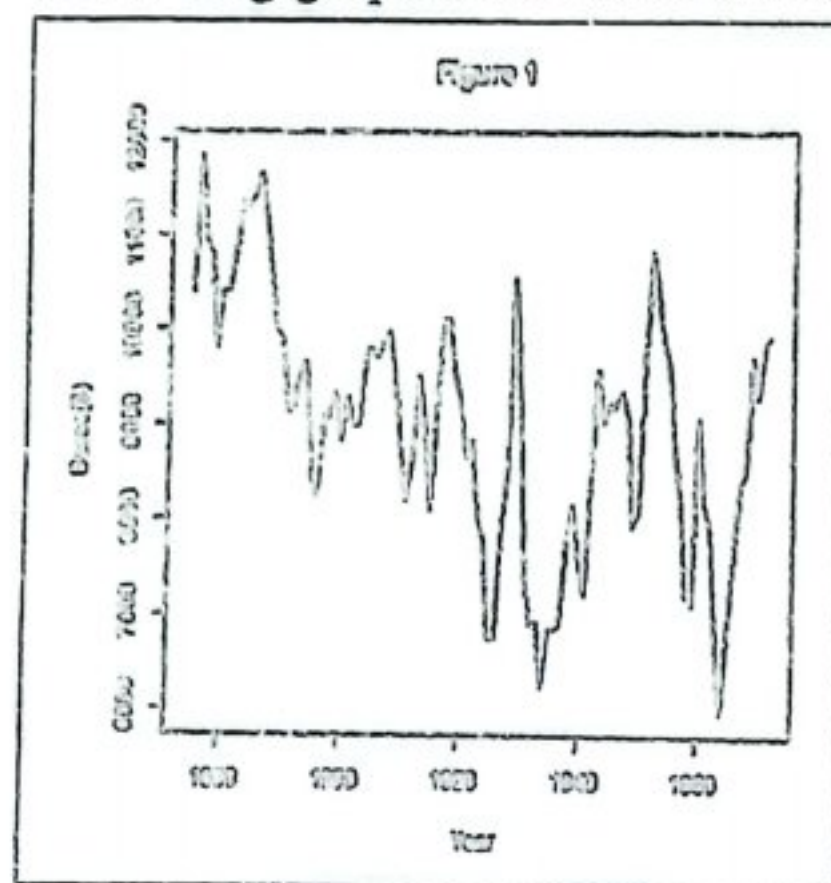
Course code: STAT-306

Course Title: Time Series Analysis

Full marks: 20

Answer the following questions.

- Q1. Write the general steps of time series modeling.
- Q2. Differentiate between
- Strong vs Weak stationary,
 - White noise & IID noise,
 - Single vs Double exponential smoothing,
 - Random walk with vs without drift.
- Q3. What is meant by the hyper parameters? Mention the use of such parameters in smoothing techniques.
- Q4. When a preliminary transformation is required for a given time series data? Explain the reasons/features with respect to a time series plot.
- Q5. Following graphs are the time series and its estimated residuals plot, respectively.



- Mention the important features of the plots.
- Describe the process to fix/quantify the features of the plots.

Best of Luck!

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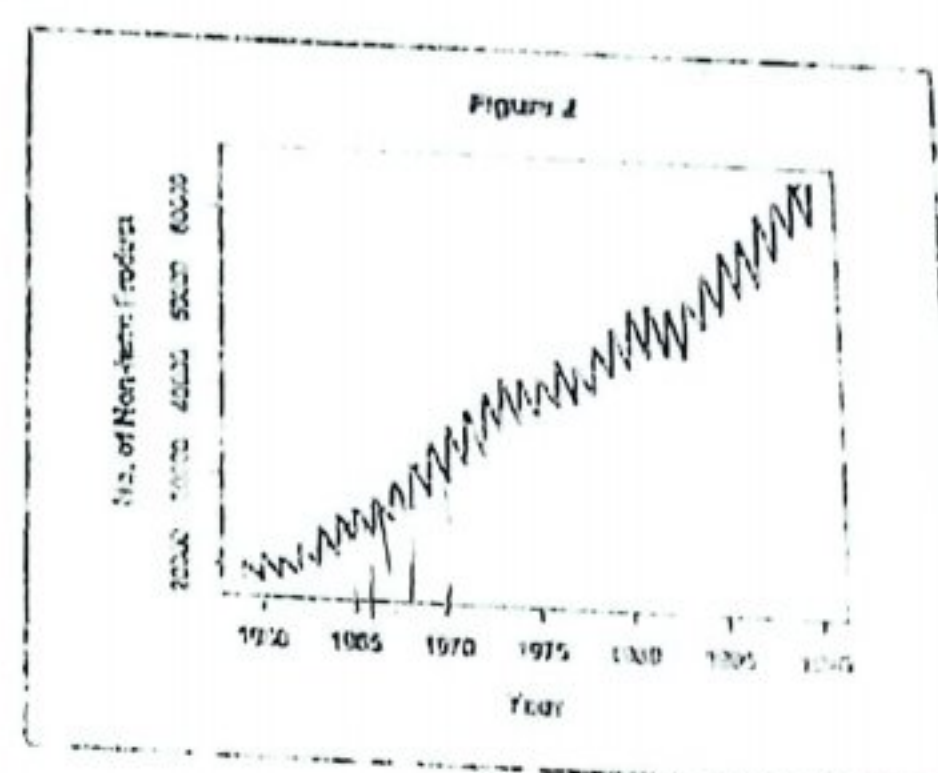
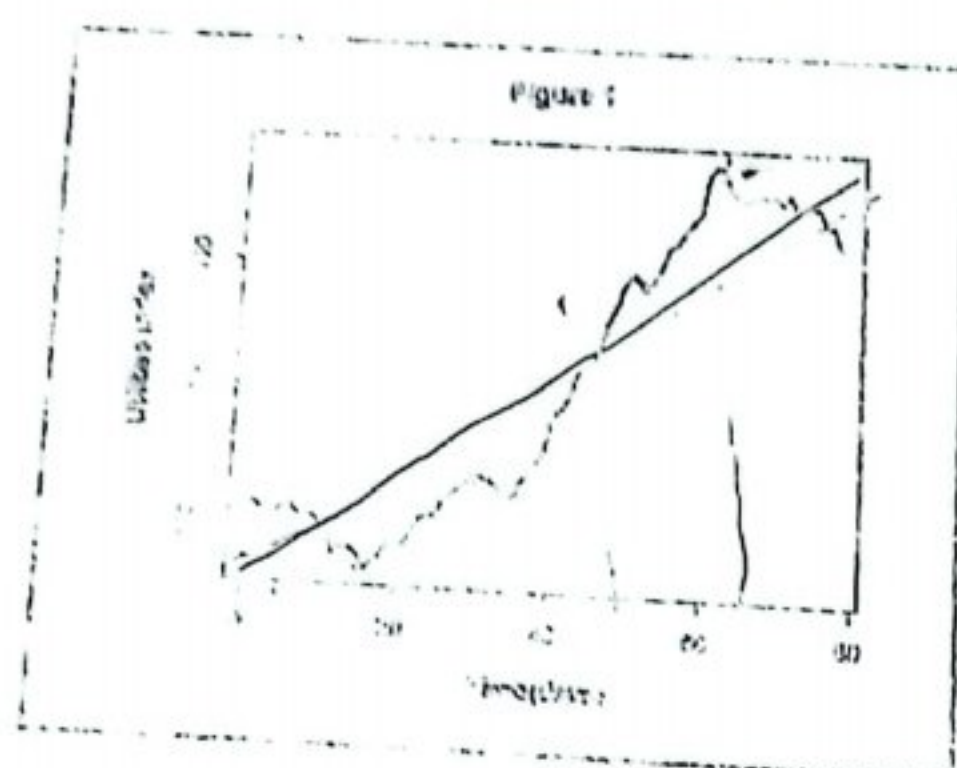
Full marks: 20

Answer the following questions.

Consider the following data sets:

Data1: The Dow-Jones Utilities Index, August 28 - Dec. 18, 1972

Data2: Australian Gross Non-farm Product at average 1984/5 prices in millions of dollars, Sep. 1959 - Dec. 1990



Q1. Compare the two time series plots, Figure 1 and 2, respectively. Your comparison should include the followings:

- Definition of time series in the context of the above two cases,
- Types of time series,
- Objectives of the above cases,
- Components of time series,
- Stationary assumption and transformation,
- Data collection,
- Models and the ways for the estimation of components (if any),
- Checking dependency among the estimated residuals.

Best of Luck!

Time series analysis:

A time series is a set of observations x_t , each one being recorded at a specific time t . In other words, arrangement of statistical data in accordance with occurrence of time is known as time series.

Time series analysis is a quantitative forecasting method to predict future values, where numerical data are obtained at regular time intervals and projections are based on past and present observations.

Examples:

- Hourly temperature readings
- Daily stock prices
- Weekly traffic volume
- Annual growth rate.

Practical Example:

Suppose, we have the data of population of the U.S.A at 10 year intervals 1790-1990. That is a time series data. Now, for analyzing this data we have to plot it. In x axis we take "Year" and in y axis we take "population". By plotting the population against time, we get a time series curve. In this curve, we find exponential trend in data. That is, the population of U.S.A increases exponentially.

Over time. And then we can find or predict the future values of population. This is a practical example of time series analysis.

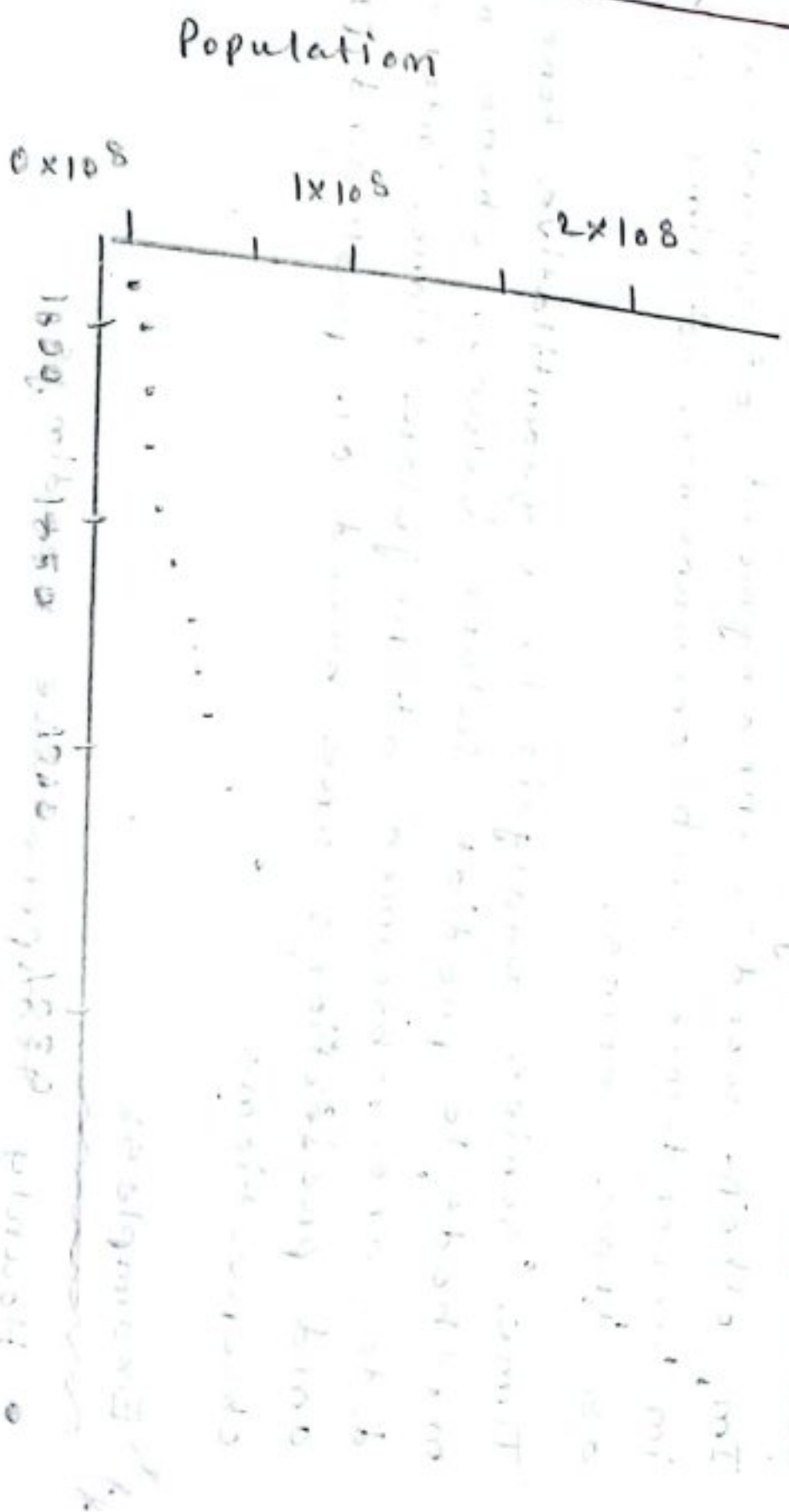


Figure 1.10

Types of time series: ***

There are two types of time series, such as

(i) Discrete time series:

A time series is said to be discrete, when observations are taken only at a specific time.

(ii) Continuous time series:

A time series is said to be continuous, when observations are recorded continuously over some time interval.

Assumptions of time series: ***

(i) The time gap between various values must be as far as possible equal.

(ii) It must consist of a homogeneous set of values.

(iii) Data must be available for a long period.

Uses of time series: ***

(i) To determine the type and nature of the variations in data

(ii) Helps in planning and forecasting

(iii) Make plans for the future

(iv) Helps to compare the actual current performance of accomplishments with the expected ones and analyze the causes of such variation, if any.

(v) To predict or estimate or forecast the behavior of the phenomenon in future which is very essential for business planning.

(vi) To compare the changes in the values of different phenomenon at different times or places etc.

Time series data:

A time series is a set of observations of the values that a variable takes at different time intervals such data may be collected at regular time intervals such as daily, weekly, monthly, quarterly etc.

Time series plot:

1) Plot

$$\begin{array}{r} 2 \overline{) 75} \end{array}$$

Plot +

read. table (MCMCMT Data Input error-9) FD

getwd() (get the current location path)

• Pinpoint method of various getter getLoc() function is used to locate the location of the data for input operation in the 32-bit location of copy paste and this etc.

for example:

gatu d c)

```
data <- read.table("wine.txt")
```

- user's command data file (input) and output copy and error file to save error messages. storage of problem create input and output files in user file as command input and output files for review easy way.

For example:

Setwd ("e:/STAT-306")

data ← read.table("volume.txt")

dim (data) [20] - for data (0 - 20) - row for column

column no. 6 data should
[value] head (data)

all data) [column name & # - data source]

class (data) [int - var data type - offset -

columns (data) - a column name

plot (data) [pdanalyse plot
omzet omzet co plot / σ_{om}
com diagram plot σ_{om}]

80, 100, 120, 140, 160, 180, 200, 220, 240, 260, 280, 300, 320, 340, 360, 380, 400, 420, 440, 460, 480, 500, 520, 540, 560, 580, 600, 620, 640, 660, 680, 700, 720, 740, 760, 780, 800, 820, 840, 860, 880, 900, 920, 940, 960, 980, 1000, 1020, 1040, 1060, 1080, 1100, 1120, 1140, 1160, 1180, 1200, 1220, 1240, 1260, 1280, 1300, 1320, 1340, 1360, 1380, 1400, 1420, 1440, 1460, 1480, 1500, 1520, 1540, 1560, 1580, 1600, 1620, 1640, 1660, 1680, 1700, 1720, 1740, 1760, 1780, 1800, 1820, 1840, 1860, 1880, 1900, 1920, 1940, 1960, 1980, 2000, 2020, 2040, 2060, 2080, 2100, 2120, 2140, 2160, 2180, 2200, 2220, 2240, 2260, 2280, 2300, 2320, 2340, 2360, 2380, 2400, 2420, 2440, 2460, 2480, 2500, 2520, 2540, 2560, 2580, 2600, 2620, 2640, 2660, 2680, 2700, 2720, 2740, 2760, 2780, 2800, 2820, 2840, 2860, 2880, 2900, 2920, 2940, 2960, 2980, 3000, 3020, 3040, 3060, 3080, 3100, 3120, 3140, 3160, 3180, 3200, 3220, 3240, 3260, 3280, 3300, 3320, 3340, 3360, 3380, 3400, 3420, 3440, 3460, 3480, 3500, 3520, 3540, 3560, 3580, 3600, 3620, 3640, 3660, 3680, 3700, 3720, 3740, 3760, 3780, 3800, 3820, 3840, 3860, 3880, 3900, 3920, 3940, 3960, 3980, 4000, 4020, 4040, 4060, 4080, 4100, 4120, 4140, 4160, 4180, 4200, 4220, 4240, 4260, 4280, 4300, 4320, 4340, 4360, 4380, 4400, 4420, 4440, 4460, 4480, 4500, 4520, 4540, 4560, 4580, 4600, 4620, 4640, 4660, 4680, 4700, 4720, 4740, 4760, 4780, 4800, 4820, 4840, 4860, 4880, 4900, 4920, 4940, 4960, 4980, 5000, 5020, 5040, 5060, 5080, 5100, 5120, 5140, 5160, 5180, 5200, 5220, 5240, 5260, 5280, 5300, 5320, 5340, 5360, 5380, 5400, 5420, 5440, 5460, 5480, 5500, 5520, 5540, 5560, 5580, 5600, 5620, 5640, 5660, 5680, 5700, 5720, 5740, 5760, 5780, 5800, 5820, 5840, 5860, 5880, 5900, 5920, 5940, 5960, 5980, 6000, 6020, 6040, 6060, 6080, 6100, 6120, 6140, 6160, 6180, 6200, 6220, 6240, 6260, 6280, 6300, 6320, 6340, 6360, 6380, 6400, 6420, 6440, 6460, 6480, 6500, 6520, 6540, 6560, 6580, 6600, 6620, 6640, 6660, 6680, 6700, 6720, 6740, 6760, 6780, 6800, 6820, 6840, 6860, 6880, 6900, 6920, 6940, 6960, 6980, 7000, 7020, 7040, 7060, 7080, 7100, 7120, 7140, 7160, 7180, 7200, 7220, 7240, 7260, 7280, 7300, 7320, 7340, 7360, 7380, 7400, 7420, 7440, 7460, 7480, 7500, 7520, 7540, 7560, 7580, 7600, 7620, 7640, 7660, 7680, 7700, 7720, 7740, 7760, 7780, 7800, 7820, 7840, 7860, 7880, 7900, 7920, 7940, 7960, 7980, 8000, 8020, 8040, 8060, 8080, 8100, 8120, 8140, 8160, 8180, 8200, 8220, 8240, 8260, 8280, 8300, 8320, 8340, 8360, 8380, 8400, 8420, 8440, 8460, 8480, 8500, 8520, 8540, 8560, 8580, 8600, 8620, 8640, 8660, 8680, 8700, 8720, 8740, 8760, 8780, 8800, 8820, 8840, 8860, 8880, 8900, 8920, 8940, 8960, 8980, 9000, 9020, 9040, 9060, 9080, 9100, 9120, 9140, 9160, 9180, 9200, 9220, 9240, 9260, 9280, 9300, 9320, 9340, 9360, 9380, 9400, 9420, 9440, 9460, 9480, 9500, 9520, 9540, 9560, 9580, 9600, 9620, 9640, 9660, 9680, 9700, 9720, 9740, 9760, 9780, 9800, 9820, 9840, 9860, 9880, 9900, 9920, 9940, 9960, 9980, 10000, 10020, 10040, 10060, 10080, 10100, 10120, 10140, 10160, 10180, 10200, 10220, 10240, 10260, 10280, 10300, 10320, 10340, 10360, 10380, 10400, 10420, 10440, 10460, 10480, 10500, 10520, 10540, 10560, 10580, 10600, 10620, 10640, 10660, 10680, 10700, 10720, 10740, 10760, 10780, 10800, 10820, 10840, 10860, 10880, 10900, 10920, 10940, 10960, 10980, 11000, 11020, 11040, 11060, 11080, 11100, 11120, 11140, 11160, 11180, 11200, 11220, 11240, 11260, 11280, 11300, 11320, 11340, 11360, 11380, 11400, 11420, 11440, 11460, 11480, 11500, 11520, 11540, 11560, 11580, 11600, 11620, 11640, 11660, 11680, 11700, 11720, 11740, 11760, 11780, 11800, 11820, 11840, 11860, 11880, 11900, 11920, 11940, 11960, 11980, 12000, 12020, 12040, 12060, 12080, 12100, 12120, 12140, 12160, 12180, 12200, 12220, 12240, 12260, 12280, 12300, 12320, 12340, 12360, 12380, 12400, 12420, 12440, 12460, 12480, 12500, 12520, 12540, 12560, 12580, 12600, 12620, 12640, 12660, 12680, 12700, 12720, 12740, 12760, 12780, 12800, 12820, 12840, 12860, 12880, 12900, 12920, 12940, 12960, 12980, 13000, 13020, 13040, 13060, 13080, 13100, 13120, 13140, 13160, 13180, 13200, 13220, 13240, 13260, 13280, 13300, 133

```
tdata <- ts(data, start = 1980, end = 1991,
             frequency = 12)
```

plot (tdata, xlab = "year", ylab = "year")

Practical 01: (colme data)

6atad (u Di/ 2nd Year/ STAT-306")

data = read.table("volume.txt")

view (data)

 $\dim(\text{data})$

colnames(data) = c("Year", "Profit")

head (data)

tail (data)

creating time series plot

```
# Creating ...  
tdata = ts(data, start = c(1980, 1), end = c(1990, 12))
```

frequency = 12)

`ts.plot = (tdata, xlab = "Year", ylab = "Sales")`

Time series analysis

TS-2

⊗ What are the different time series components? Illustrate them graphically. (6)

⇒ There are four time series components.

They are:

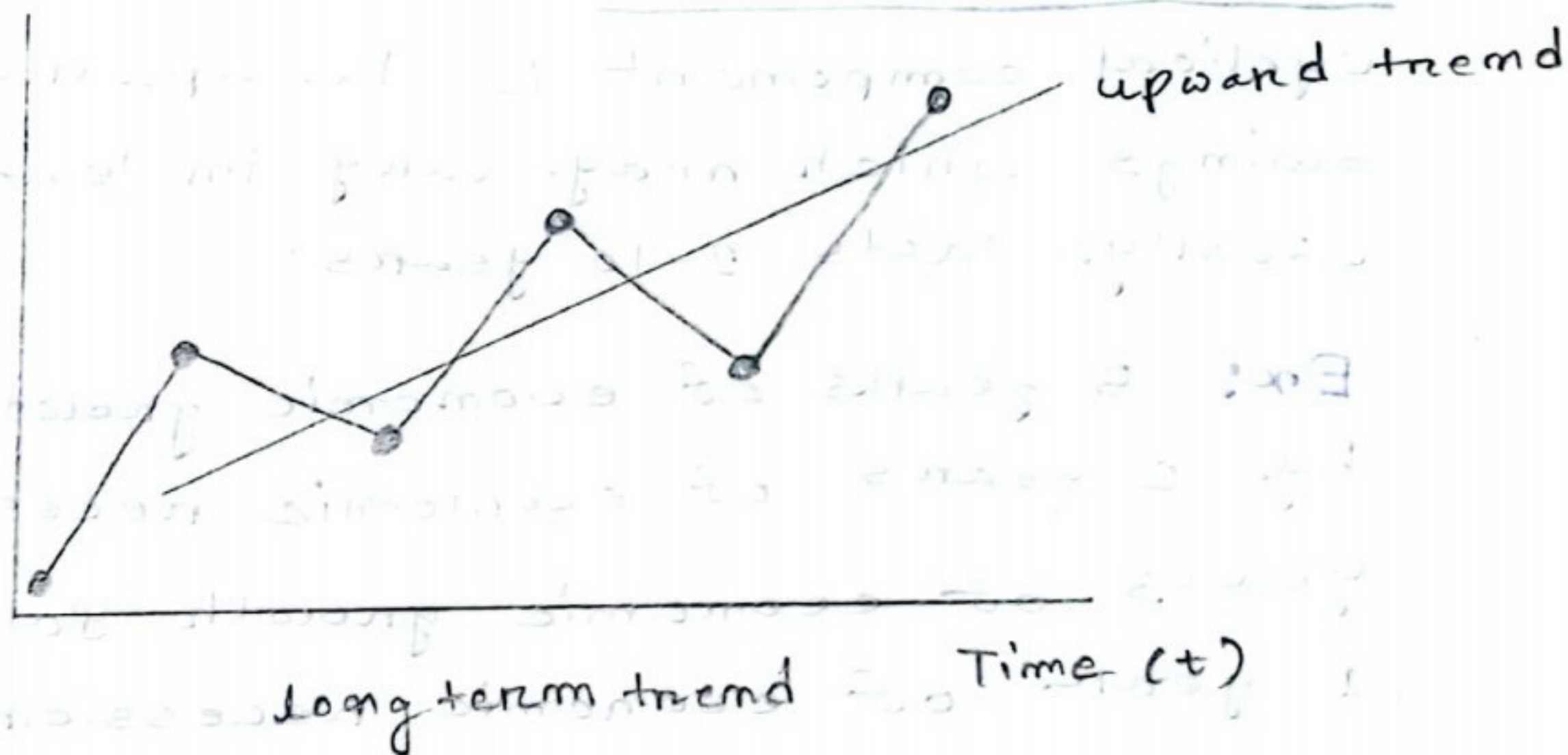
- 1) Trend component
- 2) Seasonal component
- 3) Cyclical component
- 4) Random or irregular component

① Linear trend
If we get a straight line, when the time series values are plotted on a graph, then it is called a linear trend.

Trend component:

Trend is the overall upward and downward movement in the series taken over a period of years.

Ex: population growth over the years can be seen as an upward trend.



- Population growth
- Agricultural production
- Export of manufactured goods
- Prices of consumer goods

• A definite repeated movement of a regular periodic type, occurring over short duration is known as seasonal variation.

Seasonal component:

Seasonal component is the regular patterns of upward or downward swings which is observed within a year.

Ex: Sales of ice-cream increases during summer season.



Cyclical component:

Cyclical component is the upward and downward swings which may vary in length and usually lasts 2-10 years.

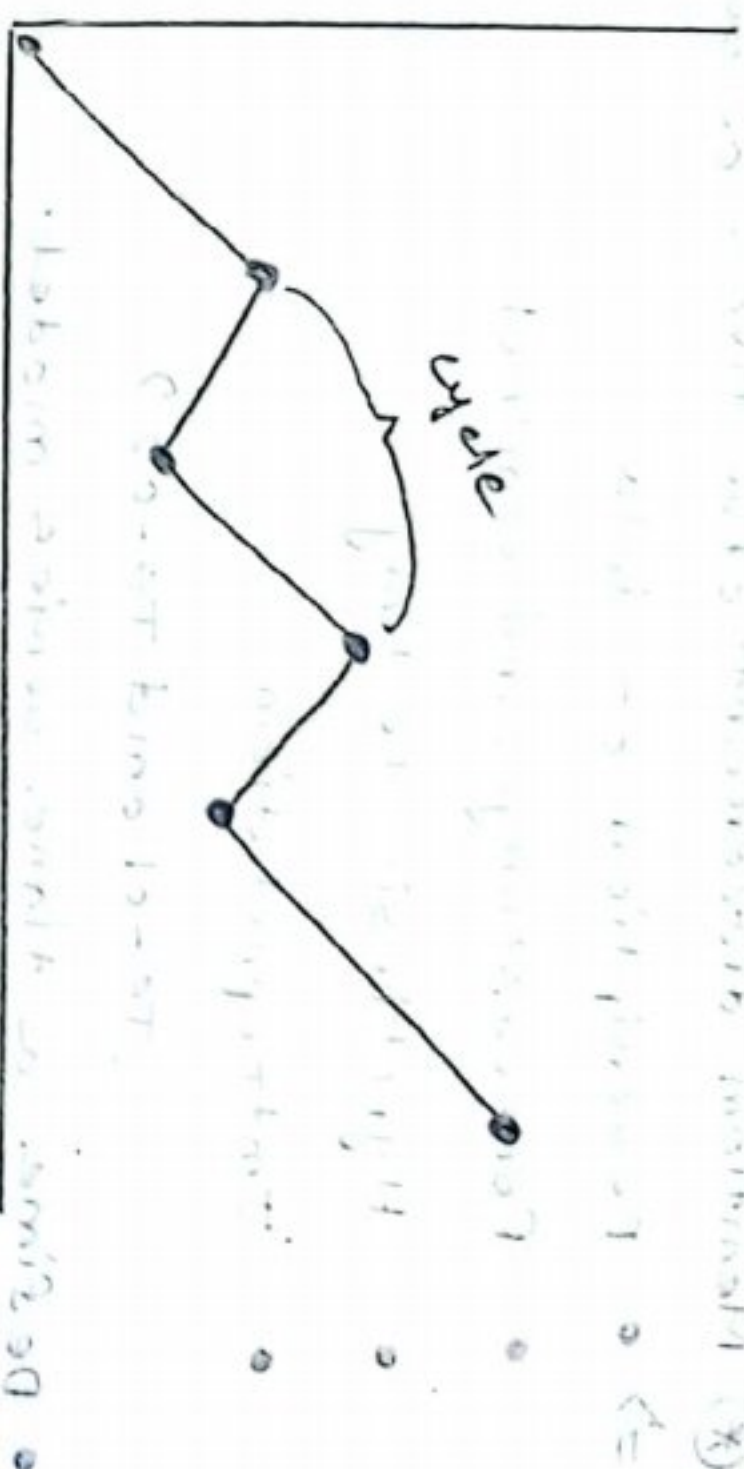
Ex: 5 years of economic growth followed by 2 years of economic recession, 3 years of economic growth followed by 1 year of economic recession.

4 phases of cyclical component are:

- Prosperity
- Recession
- Depression
- Recovery

Irregular variations are difficult to forecast because of lack of historical data. We need more than 30 years of data at least to make prediction about cycle.

$f(t)$

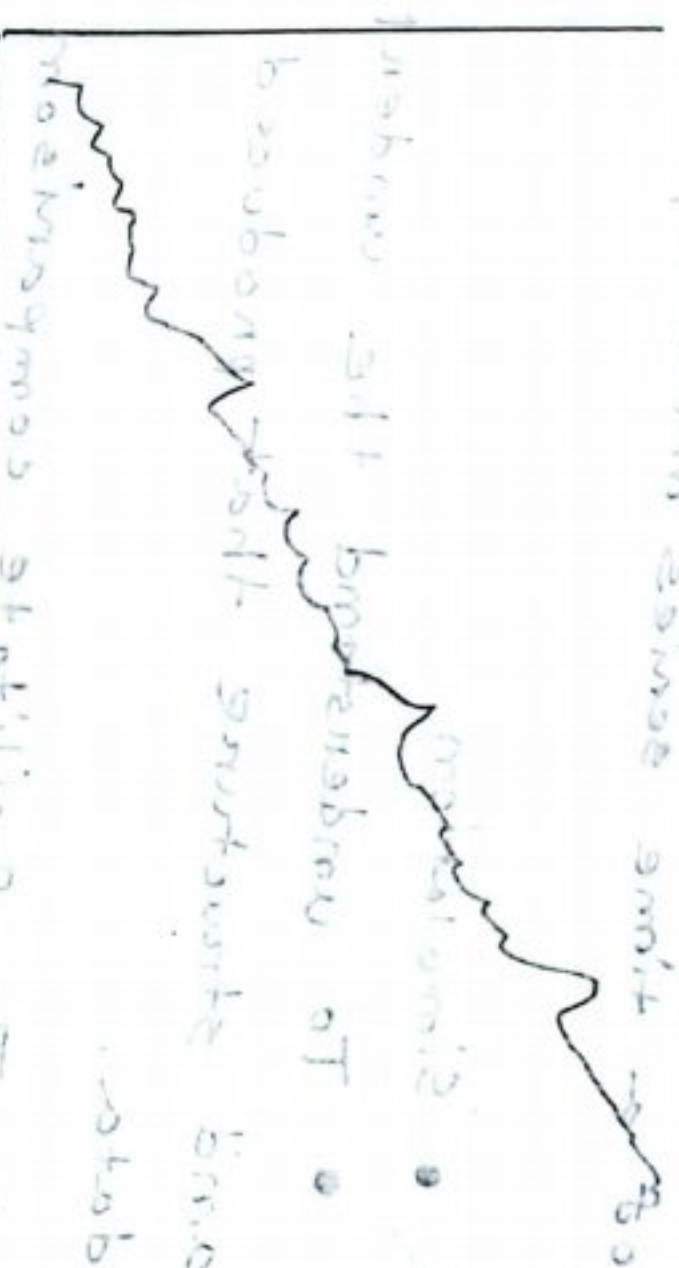


Random or Irregular component:

The variations which are happened due to uncertain and unusual cases (Natural disaster, accidents) is called random variation.

The time duration of this variation is short and it is non-repeating. It is purely random.

$f(t)$



- The irregular fluctuations cannot be predicted.
- Shows a sudden rise or fall at certain instants of time.
- Short duration and non-repeating.

* Mention different objectives of such analysis:

⇒ • Description of data

• Forecasting and control

• Hypothesis testing

• Interpretation

(TS-01 and TS-02)

• Define a time series model.

• Distinguish between a discrete time series from its continuous counterparts with suitable example.

• Discuss the general approach required for time series modeling.

• Make a comparative study among different time series components with example.

• What is meant by the second-order property of a time series model?

• Simulation

• To understand the underlying forces and structure that produced the observed data

• To facilitate comparison

• To review the current achievements

1) Define a time series model? (16)

⇒ A time series model for the observed data $\{x_t\}$ is a specification of the joint distributions of a sequence of random variables $\{x_t\}$ of which $\{x_t\}$ is postulated to be a realization.

2) What is meant by the second order property of a time series model? (16)

⇒ A complete probabilistic time series model for the sequence of random variables $\{x_1, x_2, \dots\}$ would specify all of the joint distributions of the random vectors $(x_1, \dots, x_m)$, $m=1, 2, \dots$ or equivalently all of the probabilities $P[x_1 \leq \alpha_1, \dots, x_m \leq \alpha_m]$, $-\infty < \alpha_1, \dots, \alpha_m < \infty$, $m=1, 2, \dots$

Such a specification is rarely used in time series analysis, since in general it will contain far too many parameters to be estimated from the available data. Instead we specify only the first and second order moments of the joint distributions, i.e., the expected values $E x_t$ and the expected products $E(x_{t+h} x_t)$, $t=1, 2, \dots$, $h=0, 1, 2, \dots$, focusing on properties of the sequence $\{x_t\}$ that depend only on these. Such properties of $\{x_t\}$ are referred to as second order properties.

In the particular case where all the joint distributions are multivariate normal, the second-order properties of $\{X_t\}$ completely determine the joint distributions and hence give a complete probabilistic characterization of the sequence.)

③ Discuss the general approach required for time series modeling. (15)

⇒
① Plot the series and examine the main features of the graph, checking in particular whether there is

- (a) a trend
- (b) a seasonal component
- (c) any apparent sharp changes in behaviour
- (d) any outlying observations.

② Remove the trend and seasonal components to get stationary residuals. To achieve this goal it may sometimes be necessary to apply a preliminary transformation to the data. There are several ways in which trend and seasonality can be removed.

- (i) Estimating the components and subtracting them from the data.
- (ii) Differencing the data i.e., replacing the original series $\{X_t\}$ by

$(Y_t = X_t - X_{t-d})$ for some positive integer d .

③ Choose a model to fit the residuals, making use of various sample statistics including the sample autocorrelation function.

④ Forecasting will be achieved by forecasting the residuals and then inverting the transformation described above to arrive at forecasts of the original series $\{X_t\}$.

⑤ An extremely useful alternative approach is to express the series in terms of its Fourier components.

Our goal is to predict a series that typically is not deterministic but contains a random component. Stationarity helps us to do so.

TS-03

Q1) What is meant by the stationarity of a time series? Do you think this is an important property while conducting a time series analysis? Justify your answer.

⇒ A time series has stationarity if a shift in time doesn't cause a change in the shape of the distribution.

Basic properties of the distribution (mean, variance, covariance) are constant over the time.

Yes, I think stationarity is an important property while conducting a time series analysis.

Because,

- It is easier to analyze.
- It makes the ability to understand, detect and model the data with the procedures in time series analysis.

- The model assumptions are violated when nonstationary data is used.

- Most statistical forecasting methods are based on this assumption through the use of mathematical transformations.

- Stationarity is useful concept when thinking about estimation because it suggests that something is fixed across the sequence of random variables.

It is a time series which changes from time to time. So, it is difficult to make any inferences about the time series.

Q2) Suppose, you have a time series data which shows an increasing trend but no periodic component. How could you estimate this trend from this data? Explain.

⇒ The model which contains no periodic component, the model will be of the form

$$X_t = m_t + Y_t$$

where, m_t is a slowly changing function known as the trend component and Y_t has zero mean. m_t is estimated by least squares method.

In the least squares procedure we attempt to fit a parametric family of functions e.g.,

$$m_t = a_0 + a_1 t + a_2 t^2$$

to the data $\{x_1, x_2, \dots, x_n\}$ by choosing

the parameters (a_0, a_1, a_2) to minimize

$$\sum_{t=1}^n (x_t - m_t)^2$$

- It can also be used to estimate higher order polynomial trends in the same way.

Tutorial Question - 3 (12)

① The first model is a non-stationary model. Here, it has no periodic component i.e., seasonal component and has an increasing trend.

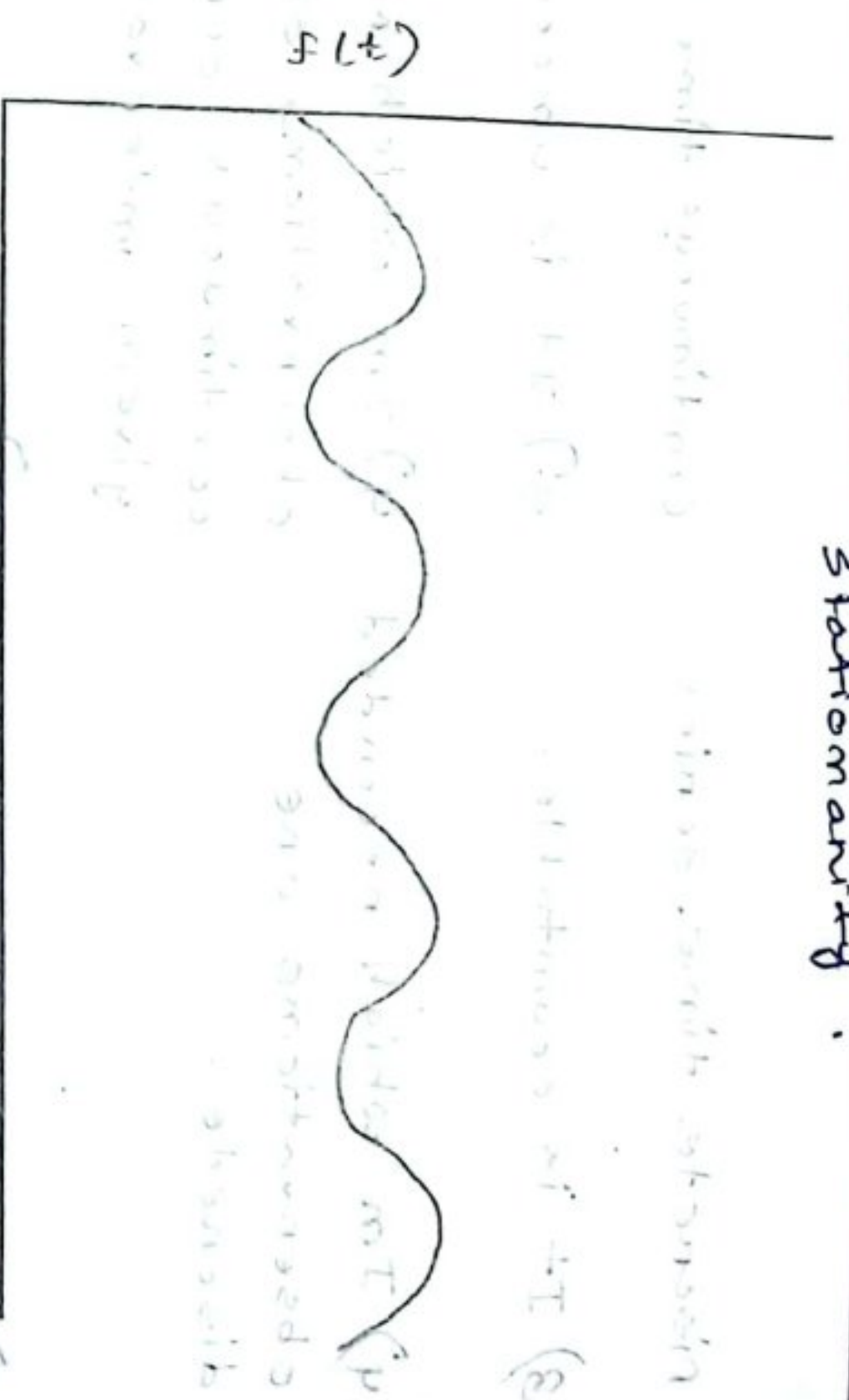
② The second model is also a non-stationary model. Here, it has both seasonal component and trend component. Trend is increasing over time. (upward)

Compare:

The two models are non-stationary. That means we have to transform these models into stationary models. For first model, we estimate trend by method of least squares. On the other hand, for second model, we use classical decomposition which is multiplicative because the seasonal component increases with time and we also use differencing.

(10)

② Sketch a time series plot which exhibits stationarity.



It is a time series plot which exhibits stationarity.

⊗ Distinguish between discrete time series and continuous time series. with suitable example. (5)

Discrete time series

① Observations are taken only at a specified time/interval over some time interval.

② It is defined over a finite domain of sequence.

Continuous time series

① Observations are recorded continuously over some time interval.

② It is defined over a finite domain of sequence.

Discrete time series		Continuous time series	
3) It is countable.		3) It is uncountable.	
4) In which recorded observations are discrete.		4) In which recorded observations are continuous for a given interval.	
5) Example: weekly stock market index		5) Example: the monthly accidental death data (1973-1978)	
(*) Make a comparative study among different time series components with example, (7)			
Trend	Seasonality	Cyclical fluctuations	Random or irregular
1) long term	1) short term	1) long and short-term	1) short-term
2) It is denoted by T_t	2) It is denoted by S_t	2) It is denoted by C_t	2) It is denoted by I_t
3) fixed time interval	3) fixed time interval	3) Not fixed time interval	3) Not fixed time interval

Trend	Seasonality	Cyclical fluctuations	Random
4) 	4) 	4) 	4) 
5) Gradual upward/downward	5) Swings between up or down	5) Repeating up and down	5) Erroneous or high fluctuation
6) Upward/downward	6) Pattern repeatable	6) No fixed period	6) Not repeatable
7) Predictable	7) Predictable	7) Challenging	7) Challenging
8) Ex: Inflation, stock prices, exchange rates etc.	8) Ex: Festivals, custom, societies, business, politics etc.	8) Ex: Monetary changes, cause of unemployment etc.	8) Ex: Natural disaster, accidents etc.

TS-04 and 05

Autocorrelation function (ACF)

- For data trend: ACF plot will exhibit slow decay as lag increases

- For data with a substantial deterministic periodic component: Plot will exhibit similar behavior with the same periodicity.

Estimation and elimination of trend and seasonality:

$$X_t = m_t + s_t + Y_t$$

classical decomposition model.

$$\text{where } EY_t = 0, s_{t+d} = s_t$$

$$\text{and } \sum_{j=1}^d s_j = 0$$

random noise component is stationary

If the seasonal and noise fluctuations appear to increase with the level of the process, then a preliminary transformation of the data is often used.

After transformation: $\ln(X)$

The transformed data does not exhibit the increasing fluctuation with increasing level that was apparent in the original data.

This suggests that the model is more appropriate for the transformed than for the original series.

Additive Decomposition:

$$Y_t = T_t + S_t + C_t + R_t$$

Assumptions:

- 1) S_t, C_t, R_t operate with equal absolute effect irrespective of the trend. (100A effect irrespective of the trend.)
- 2) C_t (and S_t) will have positive or negative values and the total of positive and negative values for any cycle will be zero. (any year)

- 3) R_t will also have positive or negative values and in the long run $\sum R_t$ will be zero.

Multiplicative Decomposition:

$$Y_t = T_t \times S_t \times C_t \times R_t$$

$$\ln Y_t = \ln T_t + \ln S_t + \ln C_t + \ln R_t$$

Assumptions:

- 1) All the values are positive.
- 2) The geometric mean of S_t in a year, C_t in a cycle and R_t in a long term period are unity.

Time series analysis

- To assess the degree of dependence in the data and to select a model for the data that reflects this, one of the important tools we use is the sample autocorrelation function (sample ACF) of the data.

- (1) What do you mean by time series?
- (2) Explain time series, cross sectional and panel data with example.
- (3) Describe various components of a time series and illustrate them graphically.
- (4) Explain the additive and multiplicative models of a time series stating clearly the assumptions and also discuss their relative merits.

- (5) What is meant by deseasonalizing a time series?
- (6) What are the different techniques to estimate and eliminate the trend component in the absence of seasonal component.
- (7) Define a time series model.
- (8) Distinguish between a discrete time series from its continuous counterpart with suitable example.
- (9) Discuss the general approach required for time series modeling.

(10) When a time series is called stationary? Why such assumption is important in time series analysis?

(11) Sketch a time series plot which exhibits stationarity.

(12) What is meant by a time series analysis? Explain with a suitable example.

(13) Mention different objectives of such analysis.

(14) Suppose, you have a time series data which shows an increasing trend but no periodic component. How would you estimate this trend from this data? Explain.

(15) Interpret the above two time series plots and hence compare.

(16) What are the basic differences between the seasonality and cyclical variation? Explain with examples.

(17) If the estimated residuals have no trend and show smoothness, how would you quantify such feature.

(18) 2019: Question (17), (22), (23) to (25) 10-27

(19) Distinguish between stationary time series and non stationary time series. State the properties of a strictly stationary process.

(20) Define the autocovariance function (ACVF) and the autocorrelation function (ACF) of a time series.

(21) State and prove the basic properties of the ACF of a time series. (20) 10-27

(22) How could you distinguish a strong stationary process from its weak stationary counterpart?

(23) What is meant by the smoothness of a time series plot? How could you handle such smoothness in case of your data? Explain the concept with the visual presentation by a graph.

(24) Make a comparative study among different time series components with example.

Tutorial-01 (TS-01 to TS-05)

(Note for exam)

TS-01 (1) What do you mean by time series?

Ans:

A time series is a set of observations x_t , each one being recorded at a specific time t .

In other words, arrangement of statistical data in accordance with occurrence of time is known as time series.

TS-01

(2) Explain time, series, cross-sectional and panel data with example.

Ans:

(1) Time series data is a collection of observations for a single subject at different time intervals.

For example: Max Temperature, Humidity, and Wind (All three variables) in New York city (single entity) collected on first day of every year (multiple intervals of time).

(2) Cross-sectional data is a collection of observations for multiple subjects at a single point in time.

For example: Max Temperature, Humidity

and Wind (all three variables)

in New York city, SFO, Boston, Chicago (multiple entities) on 11.1.2015 (single instance).

(3) Panel data is a type of data that professionals collect by observing particular variables over a period of time at a regular frequency.

For example: Panel data may comprise annual income information and the age of individuals over a nine year period.

TS-01

(3) What is meant by a time series analysis? Explain with suitable examples.

Ans:

Time series analysis is a quantitative forecasting method to predict future values, where numerical data are obtained at regular time intervals and projections are based on past and present observations.

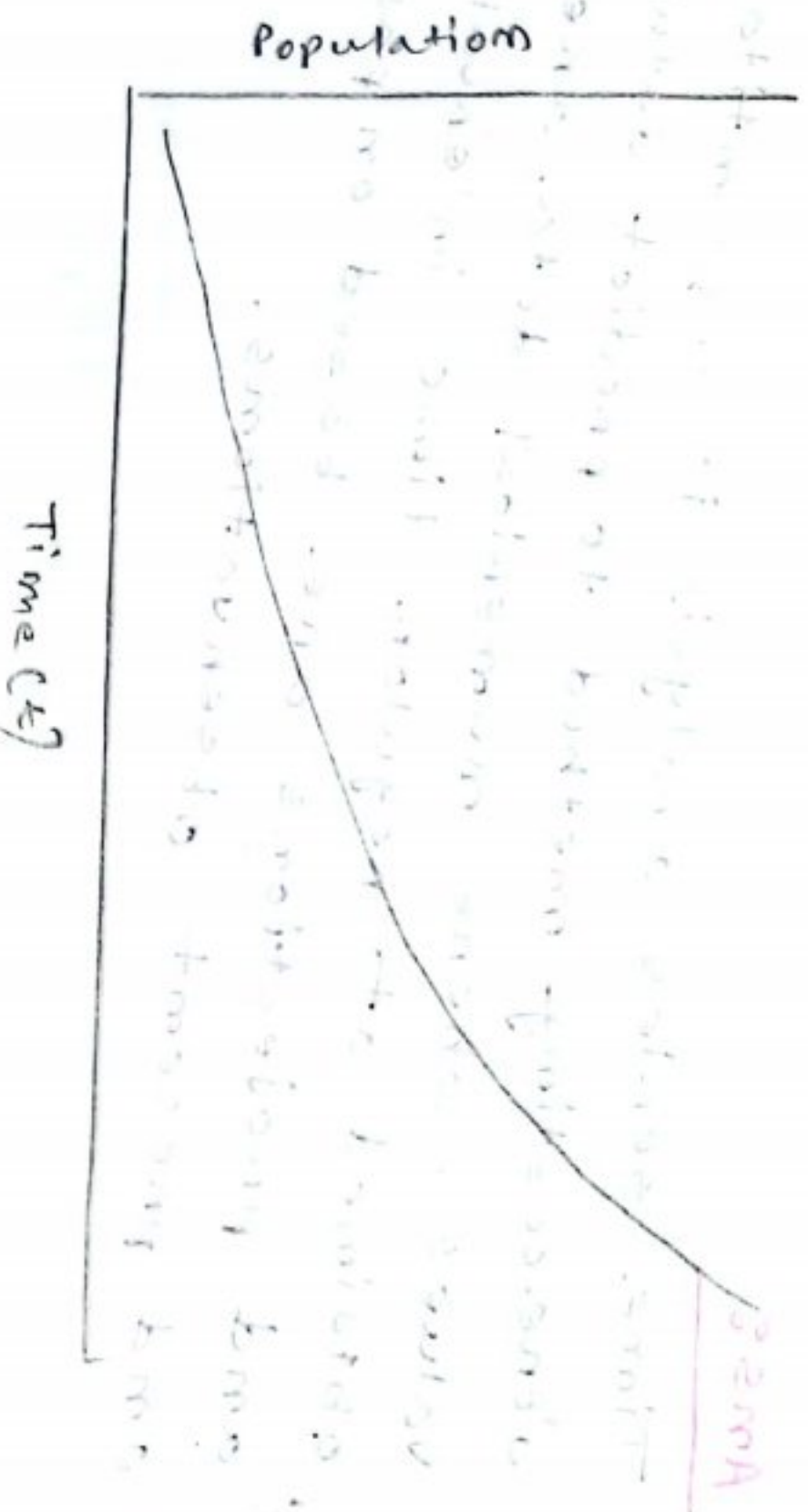
Practical Example:

Suppose, we have a data of population of the USA at 10 year intervals 1700-1990, which is a time series data.

Now, for analyzing this data, we have to plot it! In x axis we take 'Year' and in y axis we take 'population'. By plotting the population against time we get a time series curve.

In this curve, we find exponential trend in the data. That means, the population of the USA increases exponentially over time.

Then, we can find or predict the future values of the population using this information.



Other examples:

- Hourly temperature readings.
- Daily stock prices
- Weekly traffic volume.
- Annual growth rate.

(4) Mention different objectives of such analysis.

Ans:

- Description of data.
- Forecasting and control.
- Hypothesis testing.
- Interpretation.
- Simulation.
- To understand the underlying forces and structure that produced the observed data.
- To facilitate comparison.
- To review the current achievements.

Predictability of the peaks and troughs

T3-01

(5) Distinguish between a discrete time series from its continuous counterpart with suitable example.

Ans:

Already noted.

T3-02

(6) Describe various components of a time series and illustrate them graphically.

Ans:

Already noted.

T3-02

(7) Make a comparative study among different time series components with example.

Ans:

Already noted.

T3-02

(8) What are the basic differences between the seasonality and cyclical variation? Explain with examples.

Ans:

Basis of difference	Seasonal variations	Cyclical Fluctuations
1. Regularity	Both period and succession in seasonal variations are regular and their succession is definite.	Succession in cyclical fluctuations is fixed but change takes place in the period of each phase.

Basis of difference	Seasonal variations	Cyclical fluctuations
2. Period	Their period is less than one year. These changes happen during the year in a definite period.	The term in these cases is more than a year and is usually from 3 years to 10 to 12 years.
3. Causes of change	Seasonal variations are caused by changes season, traditions, fashions etc.	Cyclical fluctuations are caused by currency expansion or contraction, decrease or increase in sale, sudden change in demand etc.
4. Succession or nature of appearance	Seasonal variation is of different nature. They appear in every trade in different succession.	Cyclical fluctuations are of some nature and appear in every trade uniformly.
5. Method of measurement	They are measured in the following method: Seasonal variation index method, variable mean method,	Cyclical method is used to measure cyclical fluctuations.

Basis of difference	Seasonal variations	Cyclical fluctuations
	range, value ratio method and trend ratio method.	
6. Accuracy of measure-ment.	Seasonal variations can be measured more accurately.	Cyclical fluctuations are cannot be measured very accurately.
7. Example	Sales of ice-cream increases during summer season.	Macro-economic changes cause recession, unemployment etc.

(9) When a time series is called stationary? Why such assumption is important in time series analysis?

Ans: Already noted.

(10) Sketch a time series plot which exhibits stationarity.

Ans: Already noted.

(11) Suppose you have a time series data which shows an increasing trend but no periodic component. How could you estimate this trend from this data? Explain.

Ans: Already noted.

(12) Interpret the above two time series plots and hence compare. (Tutorial question 48)

Ans: Already noted.

(13) What is meant by deseasonalizing a time series?

Ans:

Seasonal adjustment or deseasonalization is a statistical method for removing the seasonal component of time series. It is usually done when wanting to analyse the trend, and cyclical fluctuations from the trend, of a time series independently of the seasonal components.

(14) What are the different techniques to estimate and eliminate the trend component in the absence of seasonality?

Ans:

(a) Estimation of Trend:

- 1) free hand or graphical method
- 2) Semi-average method
- 3) Smoothing

i. Smoothing with a finite moving average filter

ii. Exponential smoothing

iii. Smoothing by elimination of high frequency components.

iv. Polynomial fitting by principle of least square.

(b) Trend elimination by differencing:

This method has the advantage that it usually requires the estimation of fewer parameters and does not rest on the assumption of a trend that remains fixed throughout the observation period.

Tutorial
(2)

(15) Discuss the general approach required for time series modeling.

Ans: Already noted!

(16) Define a time series model? (ii) What is meant by the second order property of a time series model?

Ans: Already noted.

(17) Explain the additive and multiplicative models of a time series stating their assumptions and also discuss their relative merits.

Ans:

Additive model:

According to the additive model, a time series can be expressed as:

$$Y_t = T_t + S_t + C_t + U_t$$

where Y_t is the time series value at time t , T_t is the trend value at time t , S_t is the seasonal variation at time t , C_t is the cyclic variation at time t , and U_t is the random variation at time t .

Assumptions:

- i) S_t, C_t, U_t operate with equal absolute effect irrespective of the trend.
- ii) C_t (and S_t) will have positive or negative values and the total of positive and negative values for any cycle (and any years) will be zero.
- iii) U_t will also have positive or negative values and in the long run $\sum U_t$ will be zero.

Multiplicative Model:

According to the multiplicative model, a time series can be expressed as:

$$Y_t = T_t \cdot S_t \cdot C_t \cdot U_t$$

The multiplicative decomposition of a time series is same as the additive decomposition of the logarithmic values of the original time series. i.e.,

$$\log Y_t = \log T_t + \log S_t + \log C_t + \log U_t$$

Assumptions:

- i) All the values are positive.
- ii) The geometric mean of S_t in any cycle

C_t in a cycle and U_t in a long term period are unity.

Merits:

The additive model is useful when the seasonal variation is relatively constant over time.

The multiplicative model is useful when the seasonal variation increases over time.

(13) If the estimated residuals have no trend and show smoothness, how could you quantify such feature?

Ans:

Smoothness of the graph of a time series is generally indicative of the existence of some form of dependence among the observations. Such dependence can be used to advantage in forecasting future values of the series.

We quantify such dependence by the autocorrelation function as a measure of dependence.

(13) What is meant by the smoothness of a time series plot? How could you handle such smoothness in case of your data? Explain the concept with the visual presentation by a graph.

Ans:

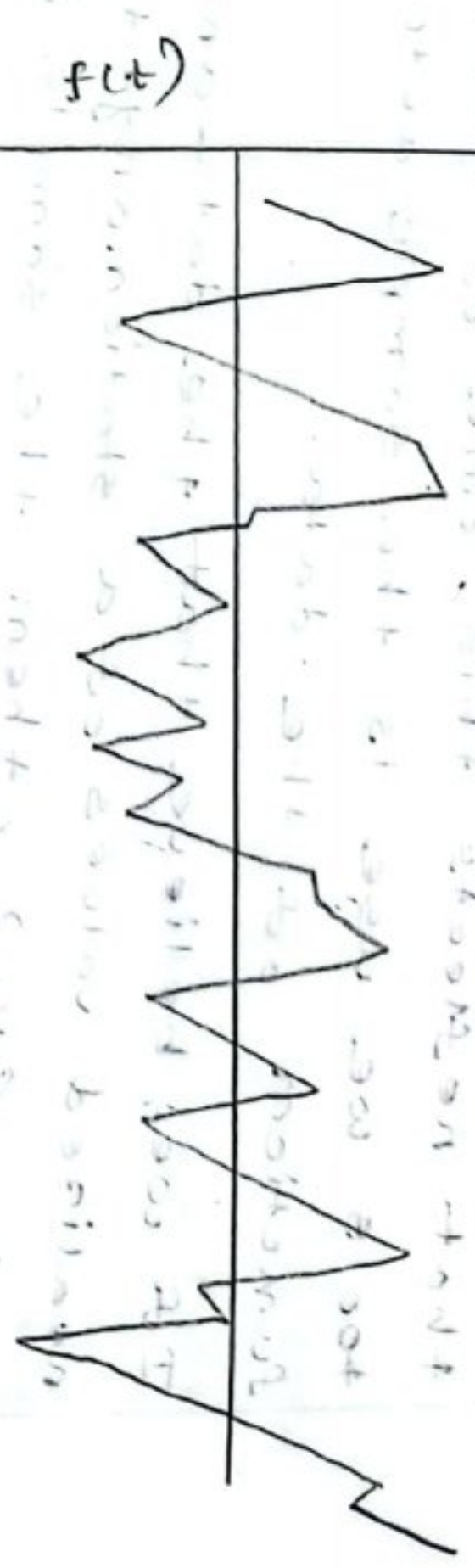


Figure: Residuals from fitting a time to the "XYZ" data

There are two interesting features of the graph of the residuals. The first is the absence of any discernible trend. The second is the smoothness of the graph.

Smoothness of the graph of a time series is generally indicative of the existence of some of dependence among the observations. This will be to our advantage, since dependence means in particular that past observations of the noise sequence can assist in predicting future values.

To assess the degree of dependence in the data and to select a model for the data that reflects this, one of the important tools we use is the sample autocorrelation function of the data.

If we believe that the data are realized values of a stationary time series $\{X_t\}$, then the sample ACF will provide us with an estimate of the ACF of $\{X_t\}$. This estimate may suggest which of the many possible stationary time series models is a suitable candidate for representing the dependence in the data.



Figure 1: The sample autocorrelation function for the $AR(1)$ residuals.

Q2 Define the autocovariance function (ACVF) and the autocorrelation function of a time series.

Ans:

Let $\{X_t\}$ be a stationary time series.

The autocovariance function of $\{X_t\}$ at lag h is

$$\gamma_X(h) = \text{cov}(X_{t+h}, X_t)$$

The autocorrelation function (ACF) of $\{X_t\}$ at lag h is

$$\rho_X(h) = \frac{\gamma_X(h)}{\gamma_X(0)} = \text{cor}(X_{t+h}, X_t)$$

Q2 state and prove the basic properties of the ACVF of a time series.

Ans:

Linearity property of covariances:

If $E X^2 < \infty$, $E Y^2 < \infty$ and $E Z^2 < \infty$ and a, b and c are any real constants,

then

$$\text{cov}(aX + bY + c, Z) = a \text{cov}(X, Z) + b \text{cov}(Y, Z)$$

Q207 Distinguish between stationary time series and non-stationary time series.

Ans:

Stationary time series	Non-stationary time series
1) A stationary time series has statistical properties on moments (mean and variance) that do not vary in time.	1) In non-stationary time series the statistical properties are changing through time.
2) 	2) 
3) Easier to analyze for long term data.	3) Much harder to analyze
4) It doesn't need any transformation.	4) It needs transformation to make it stationary.

Wm

Stationary time series

5) It is independent of time t .

Non-stationary time series

5) It is dependent of time t .

Q208 State the properties of a strictly stationary process.

Ans:

Weakly stationary:

A time series $\{X_t, t = 0, \pm 1, \pm 2, \dots\}$ is said to be weakly stationary if its mean function and covariance function is independent of time t .

Properties:

i) X_t is independent of t , (b) and

ii) X_t and X_{t+h} is independent of t for each h .

It is usually termed as simple stationary.

Strictly stationary:

$\{X_t\}$ is a strictly stationary time series if

$$(X_1, \dots, X_m)' \stackrel{d}{=} (X_{1+h}, \dots, X_{m+h})'$$

for all integers h and $m \geq 1$.

Properties:

(a) The random variables X_t are identically distributed.

(b) $(X_t, X_{t+h})' \stackrel{d}{=} (X_1, X_{1+h})'$ for all integers t and h .

(c) $\{X_t\}$ is weakly stationary if $E(X_t^2) < \infty$ for all t .

(d) Weak stationarity does not imply strict stationarity.

(e) An IID sequence is strictly stationary.

The strict stationarity property restricts the probabilistic properties of any collection of random variables to be invariant to time shifts, which implies that the probabilistic behavior of each random variable is the same across time.

Tutorial 02

Q1) How could you distinguish a strong stationary process from its weak stationary counterpart?

Ans:

Strong stationary

1) The distribution and joint distribution of a random variable is invariant to time shifting.

2) It doesn't assume any distribution, it only says that prob dist should be the same

3) It only says the probability distribution should be the same.

4) It doesn't require any moments.

5) It is also called strictly stationary.

Weak stationary

1) The mean and variance is invariant to time shift.

2) We assume normal distribution and it is fully defined by its mean and variance.

3) We define distribution by its mean and variance.

4) It requires finite and well-defined first and second moments.

5) It is also called stationary.

(25)

Final Question - 2019

(1) The time series plot of figure 1(a) shows an increasing trend with strong seasonality. There is no evidence of any cyclic behaviour here.

On the other hand, in figure 1(b), there is no seasonality. But the graph suggests the possibility of fitting a quadratic or exponential trend to the data.

The two time series plots are non-stationary.

Suggesting transformation: 1(a)

In figure 1(a) there exist trend and seasonality.

If the seasonal and noise fluctuations appear to increase with the level of the process, then a preliminary transformation of the data is often used.

Here, we see this characteristic in the plot.

So, we can use logarithmic transformation and can see that the transformed data does not exhibit the increasing fluctuation with increasing level that was apparent in the original data.

(2) Zero mean model is inappropriate for the above two cases in figure 1(a) and (b).

Because the zero-mean models have a zero-mean and constant variance and shows no predictable trend or seasonality.

Mathematical model for 1(a):

$$X_t = m_t + S_t + Y_t$$

where, m_t is a slowly changing function known as a trend component.

S_t is a function with known period referred to as a seasonal component and Y_t is a random noise component that is stationary.

Mathematical model for 1(b)

$$X_t = m + Y_t \quad ; \quad t = 1, \dots, n$$

$$\text{where, } EY_t = 0$$

(3) There are two interesting features of the graph of the residuals. The first is the absence of any discernible trend. The second is the smoothness of the graph. The smoothness of the graph of a time series is generally indicative of the existence of some form of dependence among the observations. Such dependence can be used to advantage in forecasting future values of the series. Such pattern of the residuals could be very unlikely to occur if the residuals were observations of iid noise with zero mean.

T3-01

(26) Assumptions of time series.

Ans:

- (a) The time gap between various values must be as far as possible equal.
- (b) It must consist of a homogeneous set of values.
- (c) Data must be available for a long period.

(27) Uses of time series.

Ans:

- (i) To determine the type and nature of the variations in data.
- (ii) Helps in planning and forecasting.
- (iii) Make plans for the future.
- (iv) Helps to compare the actual current performance of accomplishments with the expected ones and analyze the causes of such variation, if any.
- (v) To predict or estimate on forecast the behavior of the phenomenon in future which is very essential for business planning.

(11)

Trend elimination by differencing:

Instead of attempting to remove the noise by smoothing, we now attempt to eliminate the trend term by differencing. We define the lag-1 difference operator by ∇

$$\nabla X_t = X_t - X_{t-1} = (1-B)X_t$$

where B is the backward shift operator,

$$BX_t = X_{t-1}$$

Again,

$$\nabla^2 X_t = \nabla(\nabla X_t)$$

$$= (1-B)(1-B)X_t$$

$$= (1-2B+B^2)X_t$$

$$= X_t - 2X_{t-1} + X_{t-2}$$

If the operator ∇ is applied to a linear trend function,

$$m_t = a_0 + a_1 t$$

then we obtain the constant function

$$\nabla m_t = m_t - m_{t-1}$$

$$= a_0 + a_1 t - (a_0 + a_1 (t-1))$$

$$= a_0 + a_1 t - a_0 - a_1 t + a_1$$

$$= a_1$$

(12)

In the same way, any polynomial trend of degree k can be reduced to a constant by application of the operator ∇^k .

For example, if $X_t = m_t + Y_t$, where $m_t = \sum_{j=0}^k a_j t^j$ and Y_t is stationary with mean zero, application of ∇^k gives

$$\nabla^k X_t = \nabla^k a_k + \nabla^k Y_t$$

$$\text{where, } \nabla^k m_t = k! a_k$$

a stationary process with mean $k! a_k$

* Estimation and elimination of both trend and seasonality

① 'Classical Decomposition', in which trend and/or seasonality are estimated and subtracted from the data to generate a noise sequence.

Explication:

Classical decomposition model

$$X_t = m_t + s_t + Y_t, \quad t=1, \dots, m$$

$$\text{where } EY_t = 0, \quad s_{t+d} = s_t,$$

$$\text{and } \sum_{j=1}^d s_j = 0$$

- Suppose, we have observations $\{x_1, \dots, x_m\}$. The trend is first estimated by applying a moving average filter specially chosen to eliminate the seasonal component and to dampen the noise.

If the period d is even, say $d=2q$, then we use,

$$\hat{m}_t = \frac{(0.5x_t - q + x_{t-q+1} + \dots + x_{t+q-1} + 0.5x_{t+q})}{d}$$

where, $q < t \leq m-q$

If the period d is odd, say $d=2q+1$, then we use, (simple moving avg)

$$\hat{m}_t = \frac{(2q+1)^{-1} \sum_{j=-q}^q x_{t-j}}{d}, \quad q+1 \leq t \leq m-q$$

The second step is to estimate the seasonal component.

For each $k=1, \dots, d$, we compute the average w_k of the deviations $\{x_{k+id} - \hat{m}_{k+id}\}$, $q < k+id \leq m-q$.

Since these average deviations do not necessarily sum to zero, we estimate the seasonal component $\hat{s}_k$ as

$$\hat{s}_k = w_k - \frac{1}{d} \sum_{i=1}^d w_i, \quad k=1, \dots, d$$

$$\text{and } \hat{s}_k = \hat{s}_{k-d}, \quad k > d$$

- The deseasonalized data is then designed to be the original series with the estimated seasonal component removed i.e.,

$$d_t = x_t - \hat{s}_t, \quad t=1, \dots, m$$

- Finally, we reestimate the trend from the deseasonalized data $\{d_t\}$ using one of the methods such as polynomial fitting.

- In terms of this reestimated trend and the estimated seasonal component, the estimated noise series is then given by,

$$\hat{Y}_t = x_t - \hat{m}_t - \hat{s}_t, \quad t=1, \dots, m$$

The reestimation of the trend can be done in order to have a parametric form for the trend that can be extrapolated for the purposes of prediction and simulation.

② "Differencing" in which trend and/or seasonal components are removed from the data by repeated differencing at one or more lags in order to generate a noise sequence.

Explanation:

The technique of differencing that we applied earlier to non-seasonal data can be adapted to deal with seasonality of period d by introducing the lag- d differencing operator ∇_d defined by,

$$\nabla_d X_t = X_t - X_{t-d} = (1 - B^d)X_t$$

Applying the operator ∇_d to the model

$$X_t = m_t + s_t + Y_t$$

where $\{s_t\}$ has period d , we obtain

$$\nabla_d X_t = m_t - m_{t-d} + Y_t - Y_{t-d}$$

which gives a decomposition of the difference $\nabla_d X_t$ into a trend component

$$(m_t - m_{t-d})$$

and a noise term $(Y_t - Y_{t-d})$.

The trend, $m_t - m_{t-d}$, can then be eliminated using the methods already described, in applying a power of the operator ∇ .

*

Seasonal Variation	Cyclical variation
① Involves short term period	① Involves longer duration.
② Variations got some degree of regularity.	② No regular variations are needed.
③ Almost identical on same patterns arises every year.	③ Patterns completely depends upon market conditions.
④ helps to predict demand of the product as per upcoming times.	④ helps to formulate policies as per market i.e. if there is opportunity, how to take its benefit, but if there is threat how to avoid our weakness to overcome.
⑤ lasts almost within a year.	⑤ usually lasts 2-10 years.
⑥ There is no sequence.	⑥ A sequence is followed.

Seasonal variation

(3) It causes due to weather change, social customs, festivals etc.

Cyclical variation

(7) It causes due to 4 phases of business cycle.

(i) Prosperity (ii) Recession (iii) Depression (iv) Recovery.

(8) Example: Sales of ice-cream increases during summer season.

(8) Example: Macro economic changes cause recession and unemployment etc.

(9) * The sample autocorrelation function

To assess the degree of dependence in the data and to select a model for the data that reflects this, one of the important tools is use of the sample autocorrelation function (sample ACF) of the data. If we believe that the data are realized values of a stationary time series $\{x_t\}$, then the sample ACF

will provide with an estimate of the ACF of $\{x_t\}$. This estimate may suggest which of the many possible stationary time series models are a suitable candidate for representing the dependence in the data.

Let $x_1, \dots, x_n$ be observations of a time series. The sample mean of $x_1, \dots, x_n$ is,

$$\bar{x} = \frac{1}{n} \sum_{t=1}^n x_t$$

The sample autocovariance function is,

$$\hat{\gamma}(h) = \frac{1}{n-|h|} \sum_{t=1}^{n-|h|} (x_{t+|h|} - \bar{x})(x_t - \bar{x}), \quad -n < h < n$$

The sample autocorrelation function is,

$$\hat{\gamma}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}, \quad -n < h < n$$

Remarks:

The sample autocovariance and autocorrelation functions can be computed for any dataset $\{x_1, \dots, x_n\}$ and are not restricted to observations from a stationary time series.

- For data containing trend, $\{ \hat{S}(h) \}$ will exhibit slow decay as h increases.

- For data with a substantial deterministic periodic component, $\{ \hat{S}(h) \}$ will exhibit similar behaviour with the same periodicity.

Thus $\hat{S}(h)$ can be useful as an indicator of non-stationarity.

Sequence for First tutorial

- ① What is time series?
- ② Components of time series.
- ③ Objectives of time series.
- ④ Stationarity and importance of stationarity.
- ⑤ Model with trend in the absence of seasonality (and with seasonality) Interpretation by time series plot.
- ⑥ Estimation and elimination of trend and seasonality; Preliminary transformation
- ⑦ Choosing between additive and multiplicative decomposition.

⑧ Autocorrelation function

If we have positive autocorrelation, again we want a model. If it doesn't and it follows stationary process, then our work is done.

- The additive decomposition is the most appropriate if the magnitude of the seasonal fluctuations, or the variation around the trend-cycle, does not vary with the level of the time series.

When the variation in the seasonal pattern, or the variation around the trend-cycle, appears to be proportional to the level of the process, then a multiplicative decomposition is more appropriate.



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Sequence for First tutorial

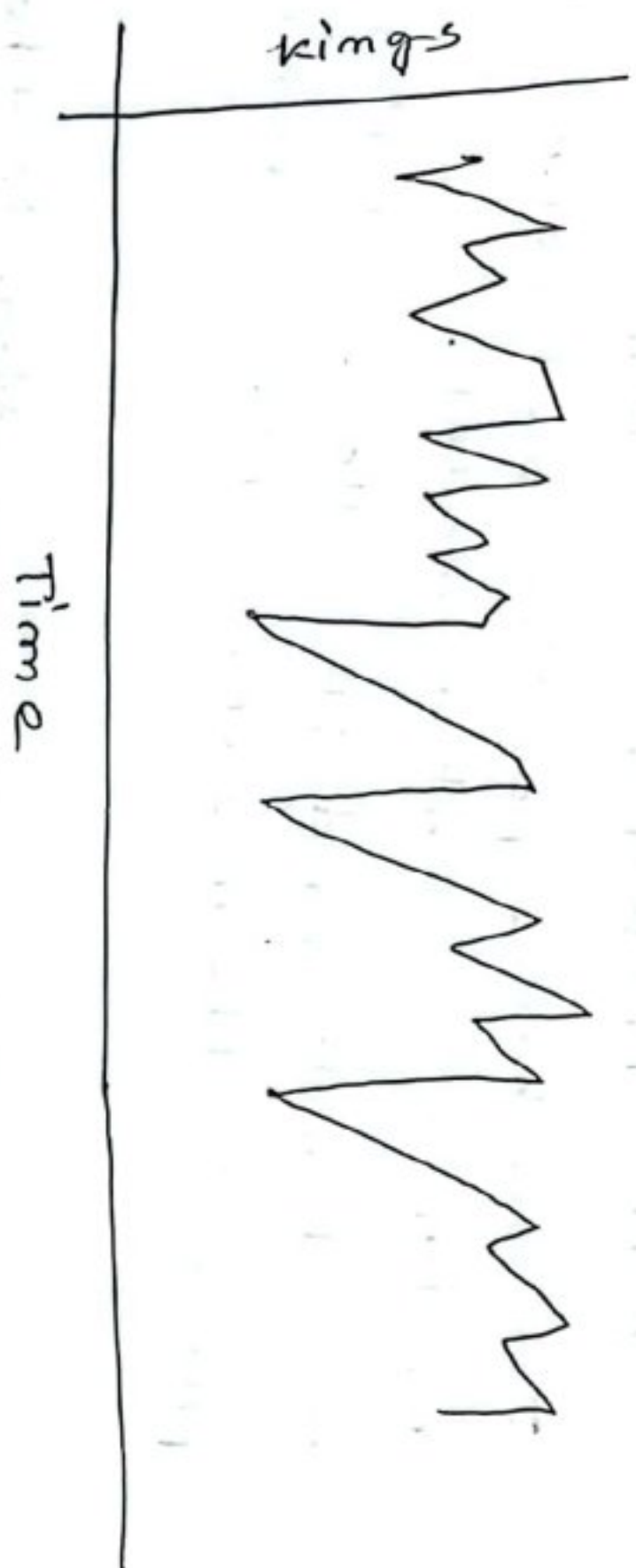
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- ⑥ Estimation and elimination of trend and seasonality; (preliminary transformation also)
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①



At this point we could guess that this time series could be described using an additive model, since the random fluctuations in the data are roughly constant in size over time.

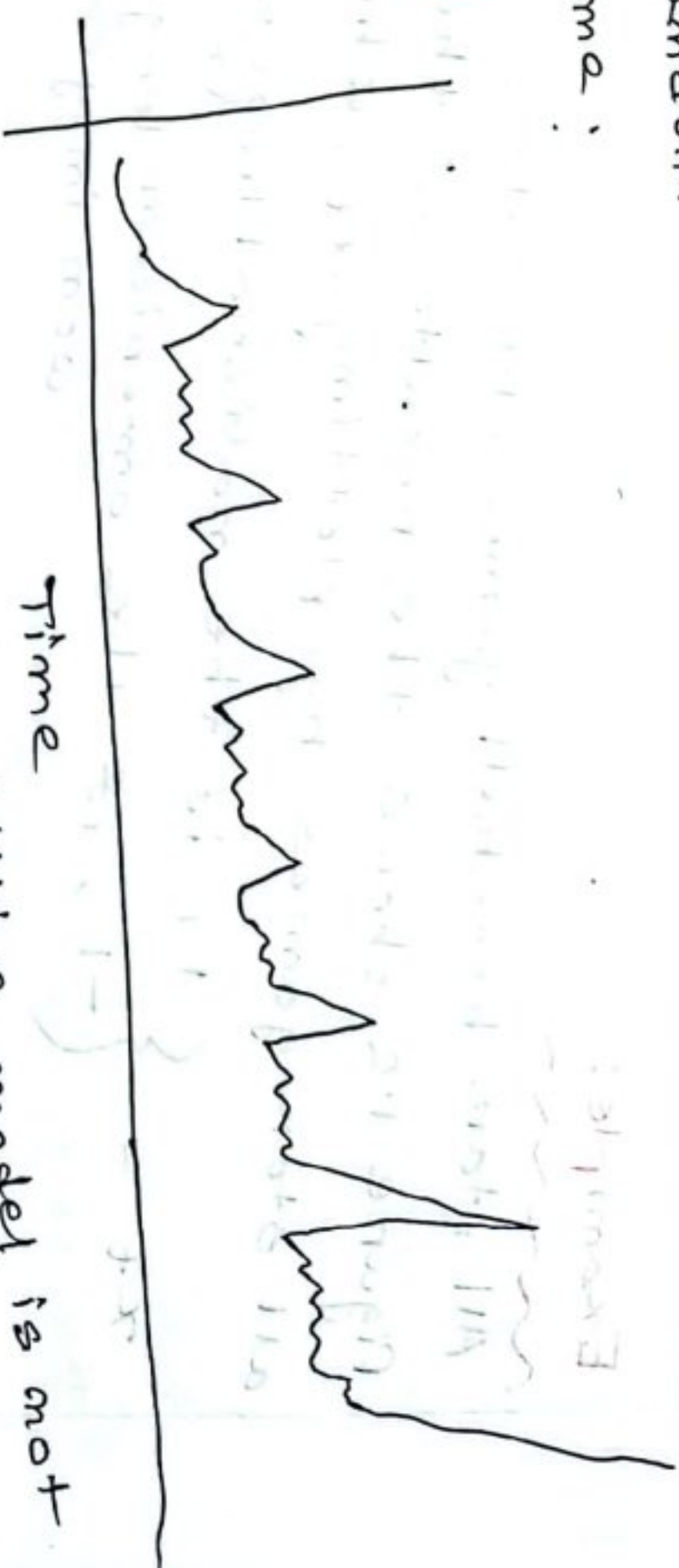
②



We can see from this time series that there is certainly some seasonal variation in the number of births per year; there is a peak in every summer and a trough in every winter.

Again it seems like this could be described using an additive model, as the seasonal fluctuations are roughly constant in size over time and do not seem to depend on the level of the time series, and the random fluctuations seem constant over time.

③



In this case, an additive model is not appropriate since the size of the seasonal and random fluctuations change over time and the level of the time series. It is then appropriate to transform the time series so that we can model the data with a classic additive model.

Discrete time series

A discrete time series is one in which the set T of times at which observations are made is a discrete set, as is the case, for example, when observations are made at fixed time intervals.

Example:

All star baseball games, 1933-1995

Figure 1.2 shows the results of the all star games by plotting x_t , where

$$x_t = \begin{cases} 1, & \text{if the national league won} \\ -1, & \text{if the american league won in year } t. \end{cases}$$



Stationary time series: $\{x_t, t=0, \pm 1, \dots\}$

Let, $\{x_t\}$ be a stationary time series.

The autocovariance function (ACVF) of $\{x_t\}$ at lag h is,

$$\gamma_x(h) = \gamma_x(h, 0) = \gamma_x(t+h, t) = \text{cov}(x_{t+h}, x_t)$$

The autocorrelation function (ACF) of $\{x_t\}$ at lag h is

$$\rho_x(h) = \frac{\gamma_x(h)}{\gamma_x(0)} = \text{cor}(x_{t+h}, x_t)$$

Here, $\gamma_x(\cdot)$: autocovariance function

$\rho_x(h)$: value of $\gamma_x(\cdot)$ at lag h .

White Noise:

(Samples could follow different probability distribution)

It is usually used to describe the behaviour of an error term in a model.

For example, when modelling stationary time series:

$$x_t = \alpha + \theta x_{t-1} + \epsilon_t$$

ϵ_t follows a white noise process
if, $\text{cov}[\epsilon_t, \epsilon_{t-i}] = 0$ if $i \neq 0$

IID noise: (special case of white noise)

- Each sample has the same probability distribution,

$$\{X_t\} \sim \text{IID}(0, \sigma^2)$$

$$\gamma_X(t+h, t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

Defn:

Let $\{X_t\}$ be a time series with $E(X_t^2) < \infty$

The mean function of $\{X_t\}$ is,

$$\mu_X(t) = E(X_t)$$

The covariance function of $\{X_t\}$ is,

$$\gamma_X(n, s) = \text{cov}(X_n, X_s)$$

$$= E[(X_n - \mu_X(n))(X_s - \mu_X(s))]$$

for all integers n and s .

Basic properties of $\gamma(\cdot)$:

$$\gamma(0) \geq 0,$$

$$|\gamma(h)| \leq \gamma(0) \text{ for all } h,$$

and $\gamma(\cdot)$ is even, i.e.,

$$\gamma(h) = \gamma(-h) \text{ for all } h.$$

Proof:

The first property is simply the statement that $\text{var}(X_t) \geq 0$.

The second is an immediate consequence of the fact that correlations are less than or equal to 1 in absolute value (on the Cauchy-Schwarz inequality).

and the third is established by observing that,

$$\gamma(h) = \text{cov}(X_{t+h}, X_t) = \text{cov}(X_t, X_{t+h}) = \gamma(-h)$$

- Autocovariance functions have another fundamental property, namely that of non-negativity definiteness.

A real valued function x defined on the integers is non-negative definite if

$$\sum_{i,j=1}^n a_{ij} (i-j)a_i > 0$$

for all positive integers n and vectors $a = (a_1, \dots, a_n)$ with real valued components a_i .

TS-08

Smoothing with a finite moving average filter:

The moving average provides us with the estimates

$$\hat{m}_t = (2q+1)^{-1} \sum_{j=-q}^q x_{t-j}, \quad q+1 \leq t \leq n-q$$

Since x_t is not observed for $t \leq 0$ or $t > n-q$ we can't use it for $t \leq q$ or $t > n-q$.

White noise:

If we doing any time series modeling project, and we want to make predictions in the future, the moment we realize that a particular time series is white noise, it means that it is impossible to forecast values for that series in the future.

- constant mean
- constant variance
- There is no autocorrelation.

(That means the value of the time series at a particular period has no relation with the values in the past time period)

It is impossible to make predictions for a white noise because, there is no patterns in our time series data.
(It's completely random)

Here, samples would follow different probability distribution.

IID noise:

The simplest model, for a time series is one in which there is no trend or seasonality and in which the observations are simply IID random variables with zero mean.

Even if IID(0, σ^2) sequence is WN(0, σ^2) but not conversely.

$$\{X_t\} \sim \text{IID}(0, \sigma^2)$$

$$\gamma_X(t+h, t) = \begin{cases} \sigma^2 & \text{if } h = 0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

[each sample has the same probability distribution]

TS-07

preliminary transformation is not original transformation. It is used to remove the jumping stages. (more stable form over)

Time series $z_t \rightarrow$ 2nd order moment finite z_t^2

Strong / Strict stationary time series

$$\{X_t : t = 0, \pm 1, \dots\}$$

x_1 probabilistic behaviour over
 x_2 x_{1+h} restricted ergodic IID
 x_3 x_t zero-220,
220-220,
or more ergodic

x_m x_{m+h}

error with IID over, analysis series IID over, interest in 2310, return over interest in loss to step forward behaviour probabilistic behaviour of univariate (change of univariate) change of univariate, or system over analysis over more informative result over univariate

X_t - Indexed by time.

complete time series model of
regression observation after regression.
parameters involve regression, mean
variance - tedious job. must
unnecessary complexity arise model
large number of parameters
model for independent
avoid error term

IID noise (no dependency)

autocorrelation matters,
but multicollinearity matters,
order remove this also, (Interpretation
use regression)

Binary process - group of 2 or less
random variable or discrete time,
or 0 or outcome regression,
no. of trials fixed - some trials to trial
probability of success and failure
discrete time series of regression

Random walk (with drift and without drift)

time series regression change of combination
of regression and error term

$$S_t = X_1 + X_2 + \dots + X_t; \quad t = 1, 2, \dots$$
$$S_0 = 0$$

$\{X_t\} \rightarrow$ IID noise

A random walk is one in which
future steps on directions cannot be
predicted on the basis of past history

③ The turning point test

Either positive or negative
 2872 Based on sample autocorrelation
 function

If T is the no. of turning points
 of an IID sequence of length n
 and we say that if $y_1, y_2, \dots, y_n$
 is a sequence of observations,
~~we say that~~ there is a turning
 point at time i , since the
 probability of a turning point
 at time i is $\frac{2}{3}$, the expected

value of T is

$$\mu_T = E(T) = \frac{2(n-2)}{3}$$

$$\sigma_T^2 = \text{var}(T) = \frac{(6n-29)}{90}$$

(for large n)

Test statistic,

$$Z = \frac{|T - E(T)|}{\sqrt{\text{var}(T)}}$$

$$T \sim N(\mu_T, \sigma_T^2)$$

• A large value $T - \mu_T$ indicates that
 the series is fluctuating more
 rapidly than expected for an IID
 sequence.

• A value of $T - \mu_T$ much smaller
 than zero indicates a positive
 correlation betw neighbouring
 observations.
 $Z > \Phi^{-1}(\alpha/2)$, H_0 is rejected

④ Difference sign test (nonparametric test)

order signs (e.g. - + - - -)

$y_1 \quad y_2 \quad y_3 \quad y_4 \quad y_5$

Here $y_2 > y_1$

$y_3 > y_4$

$y_4 > y_5$

Thus,
 The number of times the
 difference of series ($y_i - y_{i-1}$) is
 positive = 3

Kon am iid sequence?

$$\mu_3 = \frac{1}{2} \ln -17$$

$$\begin{array}{r} 95 \\ 11 \\ \hline 2 \end{array}$$

for large n , $\hat{s} \sim N(\mu_s, \sigma_s^2)$

Project Ho is CS-US) is long-range CS-US means there is a trend in the series.

⑤ The rank test. (non parametric)

1. 100 100 100

Rank assignment,
Tie among tie treatment (1939)
2nd, counting and 2nd toss for
3rd, 4th, counting and 2nd toss for
5th, 6th, 7th, 8th, 9th, 10th, 11th, 12th, 13th, 14th, 15th, 16th, 17th, 18th, 19th, 20th, 21st, 22nd, 23rd, 24th, 25th, 26th, 27th, 28th, 29th, 30th, 31st, 32nd, 33rd, 34th, 35th, 36th, 37th, 38th, 39th, 40th, 41st, 42nd, 43rd, 44th, 45th, 46th, 47th, 48th, 49th, 50th, 51st, 52nd, 53rd, 54th, 55th, 56th, 57th, 58th, 59th, 60th, 61st, 62nd, 63rd, 64th, 65th, 66th, 67th, 68th, 69th, 70th, 71st, 72nd, 73rd, 74th, 75th, 76th, 77th, 78th, 79th, 80th, 81st, 82nd, 83rd, 84th, 85th, 86th, 87th, 88th, 89th, 90th, 91st, 92nd, 93rd, 94th, 95th, 96th, 97th, 98th, 99th, 100th, 101st, 102nd, 103rd, 104th, 105th, 106th, 107th, 108th, 109th, 110th, 111th, 112th, 113th, 114th, 115th, 116th, 117th, 118th, 119th, 120th, 121st, 122nd, 123rd, 124th, 125th, 126th, 127th, 128th, 129th, 130th, 131st, 132nd, 133rd, 134th, 135th, 136th, 137th, 138th, 139th, 140th, 141st, 142nd, 143rd, 144th, 145th, 146th, 147th, 148th, 149th, 150th, 151st, 152nd, 153rd, 154th, 155th, 156th, 157th, 158th, 159th, 160th, 161st, 162nd, 163rd, 164th, 165th, 166th, 167th, 168th, 169th, 170th, 171st, 172nd, 173rd, 174th, 175th, 176th, 177th, 178th, 179th, 180th, 181st, 182nd, 183rd, 184th, 185th, 186th, 187th, 188th, 189th, 190th, 191st, 192nd, 193rd, 194th, 195th, 196th, 197th, 198th, 199th, 200th, 201st, 202nd, 203rd, 204th, 205th, 206th, 207th, 208th, 209th, 210th, 211st, 212nd, 213rd, 214th, 215th, 216th, 217th, 218th, 219th, 220th, 221st, 222nd, 223rd, 224th, 225th, 226th, 227th, 228th, 229th, 230th, 231st, 232nd, 233rd, 234th, 235th, 236th, 237th, 238th, 239th, 240th, 241st, 242nd, 243rd, 244th, 245th, 246th, 247th, 248th, 249th, 250th, 251st, 252nd, 253rd, 254th, 255th, 256th, 257th, 258th, 259th, 260th, 261st, 262nd, 263rd, 264th, 265th, 266th, 267th, 268th, 269th, 270th, 271st, 272nd, 273rd, 274th, 275th, 276th, 277th, 278th, 279th, 280th, 281st, 282nd, 283rd, 284th, 285th, 286th, 287th, 288th, 289th, 290th, 291st, 292nd, 293rd, 294th, 295th, 296th, 297th, 298th, 299th, 300th, 301st, 302nd, 303rd, 304th, 305th, 306th, 307th, 308th, 309th, 310th, 311st, 312nd, 313rd, 314th, 315th, 316th, 317th, 318th, 319th, 320th, 321st, 322nd, 323rd, 324th, 325th, 326th, 327th, 328th, 329th, 330th, 331st, 332nd, 333rd, 334th, 335th, 336th, 337th, 338th, 339th, 340th, 341st, 342nd, 343rd, 344th, 345th, 346th, 347th, 348th, 349th, 350th, 351st, 352nd, 353rd, 354th, 355th, 356th, 357th, 358th, 359th, 360th, 361st, 362nd, 363rd, 364th, 365th, 366th, 367th, 368th, 369th, 370th, 371st, 372nd, 373rd, 374th, 375th, 376th, 377th, 378th, 379th, 380th, 381st, 382nd, 383rd, 384th, 385th, 386th, 387th, 388th, 389th, 390th, 391st, 392nd, 393rd, 394th, 395th, 396th, 397th, 398th, 399th, 400th, 401st, 402nd, 403rd, 404th, 405th, 406th, 407th, 408th, 409th, 410th, 411st, 412nd, 413rd, 414th, 415th, 416th, 417th, 418th, 419th, 420th, 421st, 422nd, 423rd, 424th, 425th, 426th, 427th, 428th, 429th, 430th, 431st, 432nd, 433rd, 434th, 435th, 436th, 437th, 438th, 439th, 440th, 441st, 442nd, 443rd, 444th, 445th, 446th, 447th, 448th, 449th, 450th, 451st, 452nd, 453rd, 454th, 455th, 456th, 457th, 458th, 459th, 460th, 461st, 462nd, 463rd, 464th, 465th, 466th, 467th, 468th, 469th, 470th, 471st, 472nd, 473rd, 474th, 475th, 476th, 477th, 478th, 479th, 480th, 481st, 482nd, 483rd, 484th, 485th, 486th, 487th, 488th, 489th, 490th, 491st, 492nd, 493rd, 494th, 495th, 496th, 497th, 498th, 499th, 500th, 501st, 502nd, 503rd, 504th, 505th, 506th, 507th, 508th, 509th, 510th, 511st, 512nd, 513rd, 514th, 515th, 516th, 517th, 518th, 519th, 520th, 521st, 522nd, 523rd, 524th, 525th, 526th, 527th, 528th, 529th, 530th, 531st, 532nd, 533rd, 534th, 535th, 536th, 537th, 538th, 539th, 540th, 541st, 542nd, 543rd, 544th, 545th, 546th, 547th, 548th, 549th, 550th, 551st, 552nd, 553rd, 554th, 555th, 556th, 557th, 558th, 559th, 560th, 561st, 562nd, 563rd, 564th, 565th, 566th, 567th, 568th, 569th, 570th, 571st, 572nd, 573rd, 574th, 575th, 576th, 577th, 578th, 579th, 580th, 581st, 582nd, 583rd, 584th, 585th, 586th, 587th, 588th, 589th, 590th, 591st, 592nd, 593rd, 594th, 595th, 596th, 597th, 598th, 599th, 600th, 601st, 602nd, 603rd, 604th, 605th, 606th, 607th, 608th, 609th, 610th, 611st, 612nd, 613rd, 614th, 615th, 616th, 617th, 618th, 619th, 620th, 621st, 622nd, 623rd, 624th, 625th, 626th, 627th, 628th, 629th, 630th, 631st, 632nd, 633rd, 634th, 635th, 636th, 637th, 638th, 639th, 640th, 641st, 642nd, 643rd, 644th, 645th, 646th, 647th, 648th, 649th, 650th, 651st, 652nd, 653rd, 654th, 655th, 656th, 657th, 658th, 659th, 660th, 661st, 662nd, 663rd, 664th, 665th, 666th, 667th, 668th, 669th, 670th, 671st, 672nd, 673rd, 674th, 675th, 676th, 677th, 678th, 679th, 680th, 681st, 682nd, 683rd, 684th, 685th, 686th, 687th, 688th, 689th, 690th, 691st, 692nd, 693rd, 694th, 695th, 69

- The rank test is particularly useful for detecting linear trend in the data.

$p =$ the number of pairs (i,j)

Such that $y_i > y_i$ and $i' > i$

$$1-w, \dots, 1$$

The mean of p is

$$\frac{1}{n} = \frac{1}{n-1}$$

The variance of p is,

$$\sigma_p = \frac{n(n-1)(2n+5)}{72}$$

and that for large n ,
asymptotically $N(\mu_p, \sigma_p^2)$

p is appropriate
reject H_0 , if $|p - \text{upl}| / \sigma_p > \Phi_{1-\alpha/2}$

⑤ Fitting an autoregressive model to the data and choose the order which minimizes the AIC statistic.

$[AR(p)]$ model (ARMA)

order value might suggest that
Near to zero [white noise.]
the data is

A selected order arrival to zero
suggests that the data is white
noise

⑦ checking for normality;

Histogram, or Plot

value same until last? $\rightarrow$ no-order

[illegible]

base 0.
sum 1.

AR order minimum order 02- AIC

no. of children
white noise,

~~but~~ CH_2Cl_2 p value

concentration	40	0.05
100		

So, there may have some discrepancy, with getTest
then write migration.

AR 2-48mm base line 23pm
orig order 700m

sampling error

Exactly normal 22m from sampling
error in amount or plot generation
straight line 220r,

AR Model : Features

ANO consistent trend seasonality

seasonality

No obvious outliers.
No obvious condensation and room
appears to slowly warm up and
might be

30, the ALU, appropriate, value)

t	x_t	x_{t-1}
1	13	13
2	14	14
3	17	14
4	10	8
5	16	10
6		

Regress y_t on y_{t-1}
 a moderately strong relation

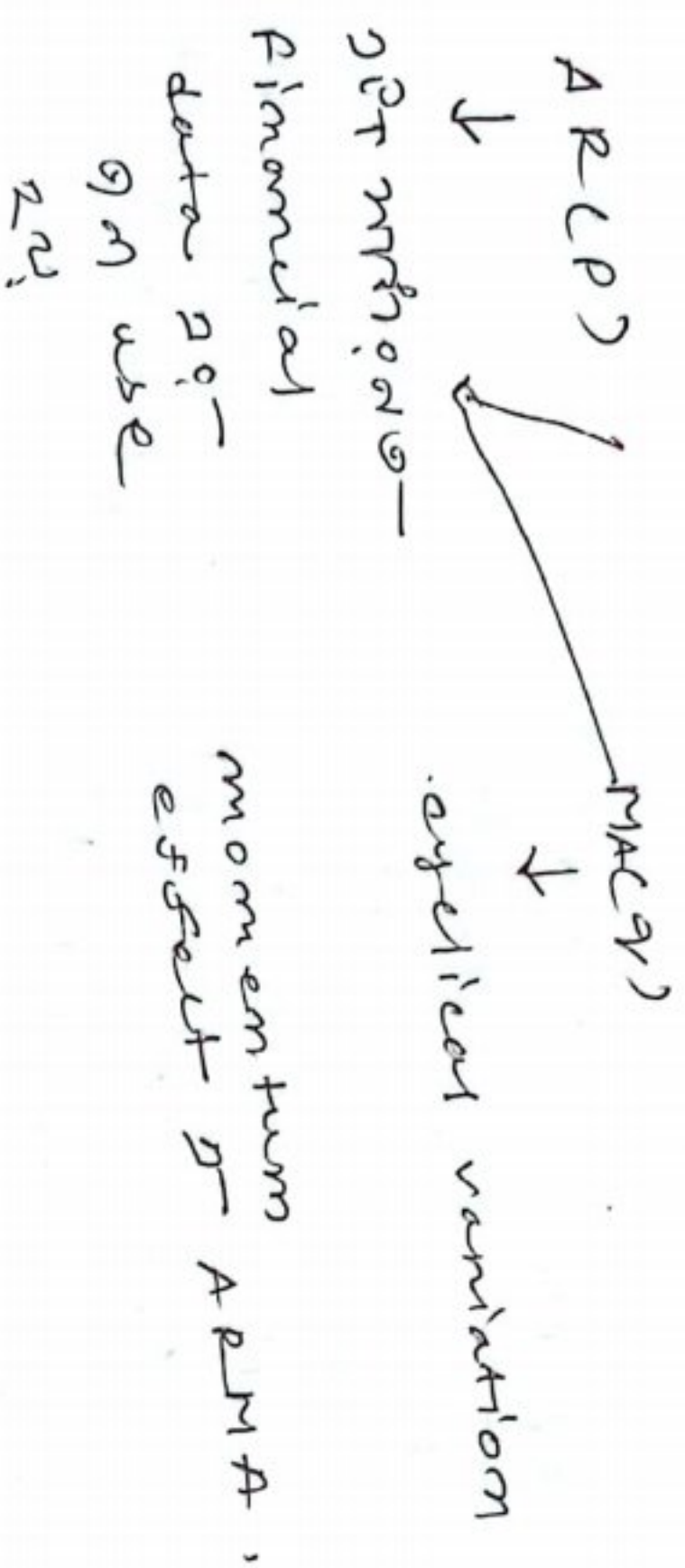
$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t$$

$$p = x + 1 + 2 + 3 + \dots + x$$

APC17 model

ARMA model

The merger between ARCP and MA(q) models.



It predicts the future values based on both the previous values and errors.

AR model error based on $2N$,

error - autocorrelation.

AR model go for preliminary model $2N$ mo, error $2N$ mo, judge mo go use $2N$ mo - AR

But modeling go to good for model is not so good for financial time series, error go feature non-1

ARMA model is a good example of the linear process.

Testing Estimated Noise Sequence!

After being find the residuals stationary, the next step is to model the estimated noise sequence.

• If there is no further modeling

TS-10 (Note)

Testing the estimated noise sequence:

After finding stationary residuals, the next step is to model the estimated noise sequence.

- If there is no dependence among between these residuals, this means that the residuals are the observations of independent random variables.
- If there is significant dependence among the residuals, then we need to look for a more complex stationary time series model for the noise that accounts such dependence.
- Dependence means that past observations of the noise sequence can assist in predicting future values.

no further modeling is required except calculating mean and variance.

H_0 : The estimated residuals are i.i.d.

(1) The sample autocorrelation function.

For large n , the sample autocorrelations of an iid sequence $y_1, \dots, y_n$ with finite variance are approximately iid with distribution $N(0, 1/n)$.

Hence, if $y_1, \dots, y_n$ is a realization of such an iid sequence, about 95% of the sample autocorrelations should fall between the bounds $\pm 1.96/\sqrt{n}$.

If we compute the sample autocorrelations up to lag h_0 and find that more than two or three values fall outside the bounds, or one value falls far outside the bounds, we therefore reject the null hypothesis.

(2) The portmanteau test

Instead of checking to see whether each sample autocorrelation $\hat{\gamma}(j)$ falls inside the bounds, it is also possible to consider the single statistic,

$$Q = n \sum_{j=1}^h \hat{\gamma}(j)^2 \sim \chi^2(h)$$

n = number of sample observations

$\hat{\gamma}(j)$ = sample autocorrelation

h = number of lags we use

If $Q > \chi^2_{(1-\alpha)}(h)$, reject H_0 .

Some modification are there, formulated by Ljung and Box (1978) in which Q is replaced by

$$Q_{LB} = n(n+2) \sum_{j=1}^h \hat{\gamma}(j)^2 / (n-j)$$

whose distribution is better approximated by chi-square distribution with h degrees of freedom.

(3) The turning point test:

If T is the number of turning points of an iid sequence of length m , then the expected value of T is,

$$\mu_T = E[T] = \frac{2(m-2)}{3}$$
$$\sigma_T^2 = \text{var}(T) = \frac{16m-29}{90} \quad [\text{variance of } T]$$

for large m ,

T is approximately $N(\mu_T, \sigma_T^2)$.

If $|T - \mu_T| / \sigma_T > \Phi_{1-\alpha/2}$, reject H_0 .

- A large value of $(T - \mu_T)$ indicates that the series is fluctuating more rapidly than expected for an iid sequence.

- A value of $(T - \mu_T)$ much smaller than zero indicates a positive correlation between neighbouring observations.

(4) The difference sign test:

For this test,

S = Number of times the difference series $y_i - y_{i-1}$ is positive. $[i = 1, 2, \dots, m]$

For an iid sequence,

$$\mu_S = E[S] = \frac{m-1}{2}$$
$$\sigma_S^2 = \text{var}[S] = \frac{m+1}{12}$$

for large m ,

S is approximately $N(\mu_S, \sigma_S^2)$

A large positive (negative) value of $(S - \mu_S)$ indicates the presence of increasing (decreasing) trend in the data.

We therefore reject the assumption of no trend in the data if

$$|S - \mu_S| / \sigma_S > \Phi_{1-\alpha/2}$$

(5) The rank test:

For this test,

p = Number of pairs (i, j) such that $y_j > y_i$ and $j > i$, $[i=1, 2, \dots, n-1]$

For an iid sequence,

$$\mu_p = \frac{1}{4} n(n-1)$$

$$\sigma_p^2 = n(n-1)(n+5)/72$$

For large n ,

p is approximately $N(\mu_p, \sigma_p^2)$

A large positive (negative) value of $(p - \mu_p)$ indicates the presence of an increasing (decreasing) trend in the data.

Reject H_0 if $|p - \mu_p| / \sigma_p > \phi_{1-\alpha/2}$

(6) Fitting an autoregressive model:

Fit an autoregressive model to the data using the Yule-Walker algorithm and choose the order that minimizes the AIC statistic. A selected order equal to zero suggests that the data is a white noise.

(7) Checking for normality:

- Histogram of Residuals
- Q-Q plot.

Autoregressive models?

Features:

- No consistent trend
 - No seasonality
 - No obvious outlier
- Difficult to judge whether the variance is constant or not

Then AR(1) model might be appropriate.

Justification: autocorrelation

Plot values of the series against lag 1 values of the series.

t	y_t	y_{t-1}
1	13	-
2	14	13
3	8	14
4	10	8
5	16	10

Regress y_t on y_{t-1} .
If it shows a moderately strong relationship, then the AR(1) model might be appropriate.

Definition:

An autoregressive (AR) model is when a value from a time series is regressed on previous values from that same time series.

For example:

$$y_t \text{ on } y_{t-1}$$

$$y_t = \beta_0 + \beta_1 y_{t-1} + \varepsilon_t$$

or $x_t = \phi x_{t-1} + z_t$

AR(1) or first order autoregressive model.

$$y_t = \beta_0 + \beta_1 y_{t-1} + \beta_2 y_{t-2} + \varepsilon_t$$

or

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + z_t$$

AR(2) or second order autoregressive model.

AR(p) model:

$$x_t = \phi_1 x_{t-1} + \phi_2 x_{t-2} + \dots + \phi_p x_{t-p} + z_t$$

where $\{z_t\} \sim WN(0, \sigma^2)$

MA(q) model:

$$x_t = \theta_1 z_{t-1} + \theta_2 z_{t-2} + \dots + \theta_q z_{t-q} + z_t$$

where $\{z_t\} \sim WN(0, \sigma^2)$

ARMA Model:

- The merger between AR(p) and MA(q) models.
 - Modelling financial time series.
 - Explains the momentum and mean reversion effects often observed in trading markets.
- AR(p) + MA(q)
- captures the shock effects, observed in the white noise terms.

— ARMA model attempts to capture both of these aspects of an AR and a MA model.

- It predicts the future values based on both the previous values and errors.

- Thus, ARMA has better performance than AR and MA models alone.

ARMA(1,1)

$$x_t - \phi_1 x_{t-1} = z_t + \theta_1 z_{t-1}$$

ARMA(p,q)

$$x_t - \phi_1 x_{t-1} + \dots + \phi_p x_{t-p} = z_t + \theta_1 z_{t-1} + \dots + \theta_q z_{t-q}$$

$\phi(z)$ and $\theta(z)$ are the p th and q th-degree polynomials,

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$$

$$\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$$

- ARMA Model is a good example of the linear process.

Exponential Smoothing:

For any fixed $\alpha \in [0, 1]$, the one-sided moving averages $\hat{m}_t$, $t = 1, 2, \dots, m$, defined by the recursions

$$\hat{m}_t = \alpha x_t + (1 - \alpha) \hat{m}_{t-1}, \quad t = 2, \dots, m \quad (1)$$

$$\text{and } \hat{m}_1 = x_1 \quad \dots (2)$$

Specify the value of α . Application of (1) and (2) is often referred to as exponential smoothing, since the recursions imply that for $t \geq 2$,

$$\hat{m}_t = \sum_{j=0}^{t-2} \alpha (1 - \alpha)^j x_{t-j} + (1 - \alpha)^{t-1} x_1$$

a weighted moving average of $x_t, x_{t-1}, \dots$ with weights decreasing exponentially (except for the last one)

Moving average vs exponential smoothing

- (1) In moving averages, the past observations are weighted equally. On the other hand, exponential smoothing assigns exponentially decreasing weights as the observation gets older.

- (2) The moving average removes noise using the mean value of data points, but exponential smoothing removes more noise from the data according to smoothing factor.

- (3) Exponential smoothing is less sensitive to outliers and noise, as it smooths them out with the weighted average than moving average.

- (4) Exponential smoothing is biased towards the past data, especially when the smoothing factor is low. This means that it may not react as quickly enough to the new information or changes in the data.

(5) Exponential smoothing may also underestimate or overestimate the forecast, depending on whether the data is trending up or down.

(6) If cyclical variations present in time series, then we use moving average.

(7) Exponential smoothing can be done using even two data points but to start the moving average we are required to set the window size which needs to be more than three for performing actual smoothing.

Random Walk

A tool called a random walk can help you understand the predictability of our time series forecast problem.

- A random walk is not a sequence of random numbers.

- It is different from a list of random numbers because the next value in the sequence is a modification of the previous value in the sequence.

- The process used to generate the series forces dependence from one time step to the next. This dependence provides some consistency from step to step rather than the large jumps, that a series of independent random numbers provides.

- Example: Price of a security on the stock market.

- We could expect a strong autocorrelation with previous observation and a linear fall off from there with previous lag values.

- A random walk is nonstationary.
 - We can make the random walk stationary by taking difference (difference in y_t) but we observe that we have no information to work here other than a series of random moves. There is no structure to learn.
 - No significant relationship between the lagged observations.
 - All correlations are small, close to zero and below the 0.5% and 99% confidence intervals beyond a few statistical studies.
 - A random walk is unpredictable. The best prediction we could make to use the observation at previous time step as what will happen in the next time step.
- Simple, the next time step will be a function of the prior time step.
(Naive forecast on a persistence model)

- Time series is a random walk if:
- It shows a strong temporal dependence that decays linearly or in a similar pattern.
 - It is nonstationary and making it stationary shows no obviously learnable structure in the data.
 - The persistence model provides the best source of reliable predictions.

$$y_t = \alpha + y_{t-1} + \epsilon_t$$

if $\alpha = 0$, then the random walk is said to be without drift.

if $\alpha \neq 0$, then the random walk is with drift.

$$y_t = y_0 + \alpha t + \sum_{i=1}^t \epsilon_i$$

$$\text{or } y_t = y_0 + \alpha t + \sum_{i=1}^t \epsilon_i, \quad t > 0$$

then, $E(y_t) = y_0 + \alpha t$

$$\text{var}(y_t) = \sigma^2 t$$

$$\text{cov}(y_i, y_j) = 0 \text{ for } i \neq j$$

The variance values are not constants but vary with time t , so this type of time series is non-stationary. Also, the mean values are constant only for random walk without drift.

$$E(y_t) = y_0$$

$$\text{var}(y_t) = \sigma^2 t$$

Deterministic trend,

$$y_t = 8t + \varepsilon_t \rightarrow E(y_t) = 8t$$

$$\text{var}(y_t) = \sigma^2$$

Stochastic trend

$$y_t = 8 + y_{t-1} + \varepsilon_t$$

$$E(y_t) = 8 + y_{t-1}$$

$$\text{var}(y_t) = \sigma^2 t$$

* $y_t = 8 + \underbrace{y_{t-1} + \varepsilon_t}_{\text{with drift}}$

$$= 8 + 8 + y_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$= 8 + 8 + 8 + y_{t-3} + \varepsilon_{t-2} + \varepsilon_{t-1} + \varepsilon_t$$

$$\therefore y_t = 8t + y_0 + \sum_{i=0}^{t-1} \varepsilon_{t-i}$$

$$E(y_t) = 8t \quad \text{var}(y_t) = t\sigma^2$$

Deterministic vs stochastic trend

* $y_t = \alpha t + \varepsilon_t$; $\varepsilon_t \sim IID(0, \sigma^2)$

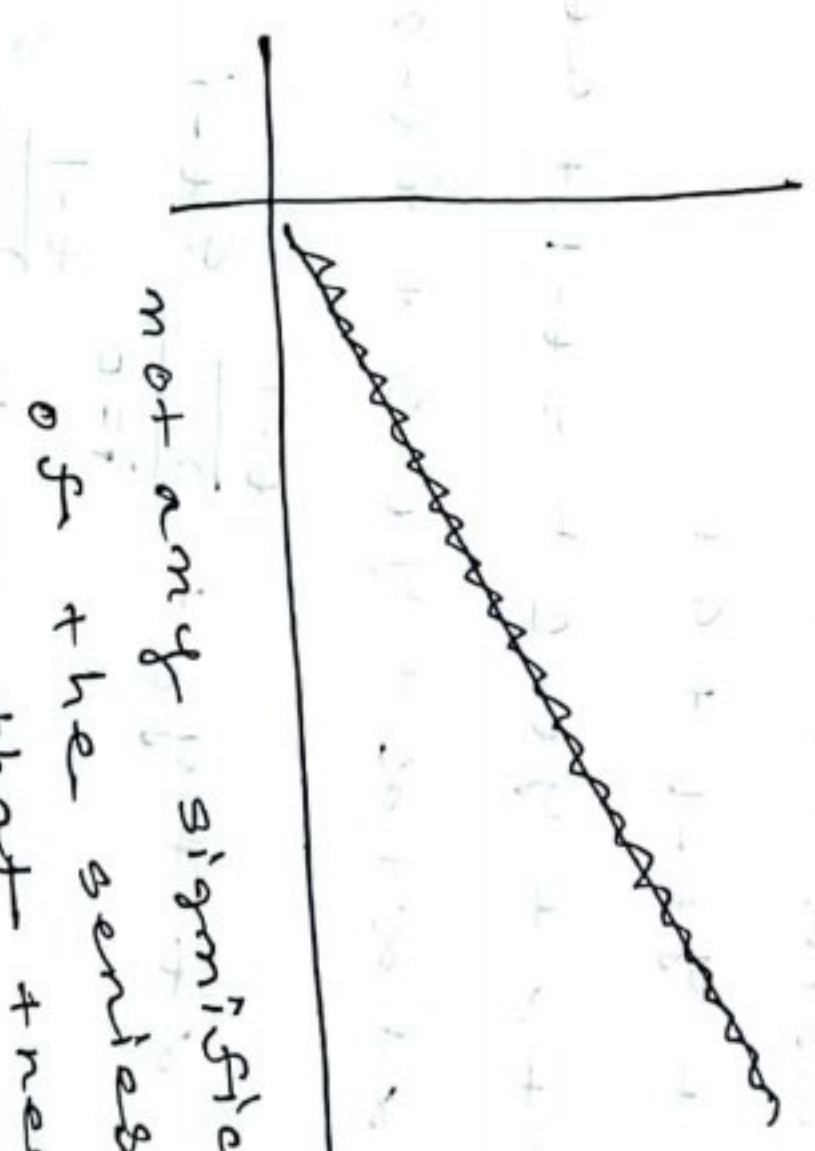
$$E(y_t) = \alpha t \quad \text{var}(y_t) = \sigma^2$$

$$y_t = \alpha + y_{t-1} + \varepsilon_t \quad [\text{just like with drift}]$$

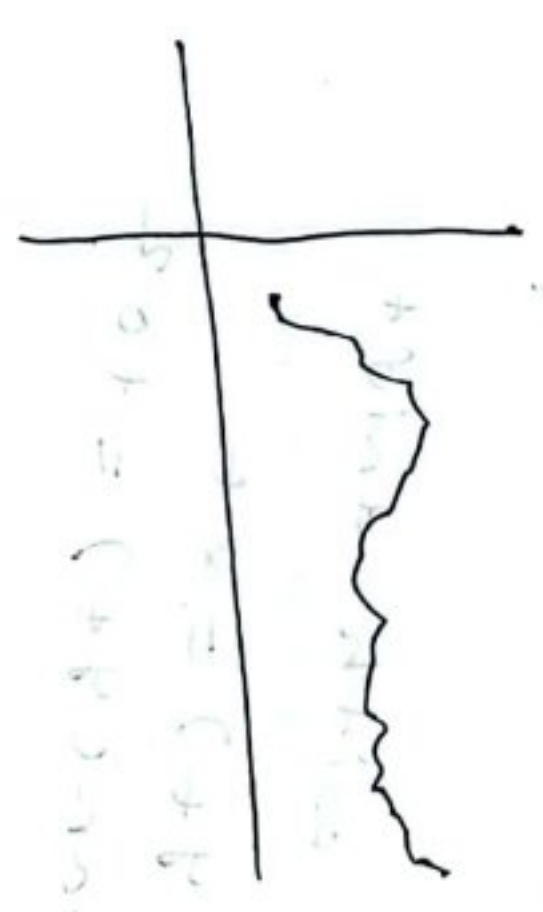
$$E(y_t) = \alpha t$$

$$\text{var}(y_t) = \sigma^2 t$$

$$y_t = \alpha t$$



not any significance deviations of the series away from that trend line.



might stay above from mean line

$$y_t = y_{t-1} + a_t$$

$$= y_{t-2} + a_{t-1} + a_t$$

$$\therefore y_t = y_0 + \sum_{i=0}^{t-1} a_{t-i}$$

$$E[y_0] = 0$$

$$\Rightarrow E[y_t] = 0$$

$$\text{var}(y_t) = t\sigma^2 = f(t)$$

Summary

Random walk with drift

$$y_t = \alpha + y_{t-1} + a_t$$

$$\Rightarrow y_t = \alpha + \alpha + y_{t-2} + a_{t-1} + a_t$$

$$\Rightarrow y_t = \alpha + \alpha + \alpha + y_{t-3} + a_{t-2} + a_{t-1} + a_t$$

$$\therefore y_t = \alpha t + y_0 + \sum_{i=0}^{t-1} a_{t-i}$$

$$\therefore y_t = y_0 + \alpha t + \sum_{i=0}^{t-1} a_{t-i}$$

Random walk with drift

$$\text{mean} = E(y_t) = \alpha t$$

$$\text{variance} = \text{var}(y_t) = t\sigma^2$$

Random walk without drift

$$y_t = y_{t-1} + a_t$$

$$\Rightarrow y_t = y_{t-2} + a_{t-1} + a_t$$

$$\Rightarrow y_t = y_0 + \sum_{i=0}^{t-1} a_{t-i} \left[y_0 + \sum_{i=1}^n a_{t-i} \right]$$

$$E(y_t) = 0$$

$$\text{var}(y_t) = t\sigma^2$$

$$\therefore E(y_0) = 0$$

Deterministic trend:

$$y_t = \alpha t + a_t, \quad a_t \sim IID(0, \sigma^2)$$

$$E(y_t) = \alpha t$$

$$\text{var}(y_t) = \sigma^2$$

Stochastic trend:

$$y_t = \alpha + y_{t-1} + a_t$$

$$E(y_t) = \alpha t \quad \text{var}(y_t) = t\sigma^2$$

Drift - unexpected and undocumented changes to data structure.

- A trend is a function of time but a drift essentially just an intercept

- ACF calculates the similarity of the values of a time series with its own lagged values. The series is lagged step by step and the correlation between the original and the shifted series is calculated. This is done up to a max lag value

- ACF calculated the covariance of a time series of x_t with itself,

— Normalised autocovariance
 PCF — measures the linear relationship between a time series and its past obs at different lags.

ACF — covariance. (how the values of a time series are related to each other as the time lag between the observations changes.

Second order property is important because,

- Help to check for stationarity in the data.
- Play a vital role in selecting appropriate models. In time series, having an accurate model allows for better forecasting performance.
- Used to estimate model parameters efficiently.
- Used to identify heteroscedasticity which refers to the varying levels of variability in a time series.
- Essential for model diagnostic checking.
- Helps to identify outliers or unusual observations in the data.

3rd Tutorial

TS-11

E17

- ARMA for linear process with drift
- context - No consistent trend
No seasonality
- stationary and order stationary process
[and order property - mean and variance should be finite]
- linear process with drift and order stationary transform into linear process with drift, drift deterministic component (trend, seasonality) for subtraction

- Decompose into - reform error,

- World's first decomposition proposed
Burgess, first order (stationary process for error linear process or constant error only)
[with order and order property for hold only]

$$X_t = \sum_{j=-\infty}^{\infty} \psi_j Z_{t-j} \quad \text{for all } t$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and

ψ_j is a sequence of constants with

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$$

Time series with drift and order stationary component
or linear association or order stationary
error, [trend and seasonal component]

we can use raw data set for modeling without applying any treatment.

• ARIMA

Integrated - drift and order stationary feature
↓
Differencing

- upward trend [possibly long term]
- seasonality
- No outlier

2nd model for least error fit error, order selection for with different error only, error error series for differenced already error, 2nd order raw data set for common error - treatment - error modeling error - error,

- AR(p) ARMA(p, q)
- MA(q) ARIMA(p, d, q)

d = differencing

Advantages with differencing error - error

- How to select the order?

- Ordinary regression - differencing error, least squared error,

- Error option for error error? error?

- ARMA(1,1) solution exists if $\phi \neq \pm 1$ (AR)

- ARMA(1,1) solution exists if $\phi \neq \pm 1$ (AR)

- ARMA(1,1) solution exists if $\phi \neq \pm 1$ (AR)

- Two sample proportion test or z-test
- (0,1).
- AR model - No trend
No seasonality
No obvious outlier
 - MA model - cyclical variation removed.
 - ARMA model - No trend
No seasonality
No obvious outlier
 - ARIMA model - Both trend and seasonality
Increasing variation
removed or strong autocorrelation
No obvious outlier
 - d = use if transformation required
diff-order technique ARMA for push over
another (pushing parameter not the
model parameters)
 - d use for parameter adjustment
change in var value change etc!
 - AIC - ARMA or GJ-concurrent ARIMA
selection
 - main - ARIMA complex model
[d use before moving ARMA even
though, can work & give its common feature
of model fit]

- model givm - Error do gnd
 - In order to get AIC least sq, 2^o order find ARMA & 3^o select 2^o,
[look for mle parameter]
[set of parameters go gnd]_maximum LE
lot of pairs of p and q must
 - H.W → code for select the order of ARIMA
 - converge algo - problem require conver-
as efficient as can be
- Ques 1
- Suppose, your data - convergent -
data set ARMA do gnd - suitable /
which model do you prefer?
find out the suitable pairs of
p and q? How would you proceed.
- ⇒ AIC check against AIC value
for 2^o or 3^o against AIC
gives plot log AIC
AIC — x axis logarithmic scale
p, q plot — point where min

- $1/\phi < 1$, then the unique stationary solution is,

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j} \quad \{X_t\} \text{ causal}$$

- Backshift operator?

$$\bullet \quad 1 \varphi 1 > 1 \quad ; \quad X_t = \cancel{Z_t} + \chi(B) \theta(B) Z_t \\ = -\theta \varphi^{-1} Z_t - \sum_{j=1}^{\infty} \varphi^{-j-1} Z_{t+j}$$

Ts-11 [22]

- cannot inappropriate situation or scenario

- stationarity check!
- portmanvar test (Ljung-Box)

stationary

- plot, p value, t0, t1 report etc
- appropriate situation & appropriate model will give you the approximate exact result

- Unit and - agreement 1. Unit not test

- Unit root test and cointegration process, vector autoregressive model, VAR, cointegration and error correction model, [autodownload]

- Add test [augmented Dk Fular test]
 H_0 : series is non-stationary.
 H_1 : series is stationary

- If $101 < 1$, $27m$ ARMA(1,1) process is invertible $27d$

invertible and
[maximum likelihood function is used
solution exists and, not and series
on function or invertible and]

$\{x_n\}$ - mean and variance are $\neq 0$ then

variance $\neq 0$ etc.
matrix of determinant $\neq 0$ variance
invertible, ρ_{21} depend

$$\text{variance } \sigma^2 \text{ zero, } \sum_{j=1}^{\infty} (-\theta)^{j-1} x_{t-j}$$

- Inventibility theorem and proof,
common van-covon motion inventible
an arm system of limb can earn
no common solution exist in.

u u

- If $| \theta | > 1$ random noninvertible (units variance covariance matrix singular)

$$Z_t = -\phi \theta^{-1} X_t + (\theta + \phi) \sum_{j=1}^{\infty} (-\theta)^{-j-1} X_{t+j}$$

- Unit circle $z=1$ point greater decline over 24 in terms of moving average, $\phi \theta^{-1} | \theta | < 1$ random point greater $\phi \theta^{-1} | \theta | > 1$ random unit root $\theta = 1$ stationary.

$$X_t = 0.5 X_{t-1} + Z_t + 0.4 Z_{t-1}$$

$$\{Z_t\} \sim WN(0, \sigma^2)$$

$$\phi(z) = 1 - 0.5z$$

$$\theta(z) = 1 + 0.4z$$

$$z = 2 > 1$$

which is located outside the unit circle. z^2 process for unit process $z=1$ causal process.

- Uniqueness and causality $\theta(z) \neq 0$ for $|z| \leq 1$
- $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p \neq 0$

MA

$$\theta(z) = 1 + 0.4z$$

$$\therefore z = -2.5 \text{ (outside)}$$

Invertibility

$$\theta(z) = 1 + 0.1z + \dots + 0.9z^9 \neq 0 \text{ for } |z| \leq 1$$

- **Difference betw A_{AR} and A_{MA}**

$\Rightarrow$ A_{AR} successive dependency

where pairwise dependency

A_{AR} is defined as the sequence of

covariances of a stationary process

- If the $\{X_t\}$ process is weakly

stationary, the covariance of X_t

and X_{t+k} depends only on the

lag k . (A_{AR})

(A_{MA})

- correlation betw X_t and X_{t+k}

TS-11 [Last class]

- AR, MA and ARMA processes

Calculation of ACF

The ACF of the causal ARMA (p,q) process is defined by

$$\phi(B)X_t = \theta(B)Z_t, \quad \{Z_t\} \sim WN(0, \sigma^2)$$

where: $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$

and $\theta(z) = 1 + \theta_1 z + \dots + \theta_q z^q$

- AR or - causality "Necessary"

- MA - Invertibility "

causal and some special expressions

causal process or get solution

$$X_t = Z_t + \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$$

$$\gamma_X(h) =$$

- ARMA process with autocovariance structure exist and AR, MA, ARMA process also causal means invertible

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2$$

- ACF for ARMA expected value

1st method

The causality assumption

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \dots (i)$$

In the special case where $\{X_t\}$ is

the linear process

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2 \dots (ii)$$

from (i) and (ii)

$$\gamma(h) = E(X_{t+h}, X_t) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+h}$$

Example:

The ARMA(1,1) process

$$X_t = Z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} Z_{t-j}$$

The autocovariance function for a

causal time series is

$$\gamma(h) = \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j+|h|+1}$$

where, $\psi_j = (\phi + \theta) \phi^{j-1}$

Case h=0

$$\text{cov}(x_t, x_t) = E(x_t x_t) = \gamma(0) = \sigma^2 - \sum_{j=0}^{\infty} \phi^j \sigma^2$$

$$\begin{aligned} E(x_t x_t) &= E\left[\left(z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} z_{t-j}\right)^2\right] \\ &= E[z_t^2] + E\left[\left((\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} z_{t-j}\right)^2\right] \\ &\quad + 2(\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} E(z_{t-j} z_t) \end{aligned}$$

since z_t is a white noise,

$$E(z_{t-j} z_t) = 0$$

$$\therefore E(x_t x_t) = E[z_t^2] + (\phi + \theta)^2 \sum_{j=1}^{\infty} \phi^{2j-2} E[z_{t-j}^2]$$

$$\begin{aligned} &= \sigma^2 + (\phi + \theta)^2 \sum_{j=1}^{\infty} \phi^{2j-2} \sigma^2 \\ &= \sigma^2 \left[1 + (\phi + \theta)^2 \sum_{j=1}^{\infty} \phi^{2j-2} \right] \\ &= \sigma^2 \left[1 + (\phi + \theta)^2 \left\{ 1 + \phi^2 + \phi^4 + \dots + \dots \right\} \right] \\ &= \sigma^2 \left[1 + \frac{(\phi + \theta)^2}{1 - \phi^2} \right] \end{aligned}$$

where $\sum_{j=1}^{\infty} \phi^{2j-2} = \sum_{j=0}^{\infty} \phi^{2j}$ is geometric series converging to $\frac{1}{1 - \phi^2}$

Case h=1

$$\begin{aligned} \text{cov}(x_t, x_{t-1}) &= E[x_t x_{t-1}] = \gamma(1) \\ &= \sigma^2 \sum_{j=0}^{\infty} \phi^j \phi^{j+1} \end{aligned}$$

$$\begin{aligned} E(x_t x_{t-1}) &= E\left[\left\{ z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} z_{t-j} \right\} \left\{ z_{t-1} + (\phi + \theta) \sum_{j=2}^{\infty} \phi^{j-2} z_{t-j} \right\}\right] \end{aligned}$$

$$= E\left[\left\{ z_t + z_{t-1}(\phi + \theta) + (\phi + \theta) \sum_{j=2}^{\infty} \phi^{j-1} z_{t-j} \right\} \left\{ z_{t-1} + (\phi + \theta) \sum_{j=2}^{\infty} \phi^{j-2} z_{t-j} \right\}\right]$$

$$\left\{ z_{t-1} + (\phi + \theta) \sum_{j=2}^{\infty} \phi^{j-2} z_{t-j} \right\}$$

$$= E\left[\left\{ z_t + z_{t-1}(\phi + \theta) + (\phi + \theta) \sum_{j=2}^{\infty} \phi^{j-2} z_{t-j} \right\} \cdot \left\{ z_{t-1} + (\phi + \theta) \sum_{j=2}^{\infty} \phi^{j-2} z_{t-j} \right\}\right]$$

$$= (\phi + \theta) E[z_t z_{t-1}] + (\phi + \theta)^2 \phi \sum_{j=2}^{\infty} \phi^{2j-4} E[z_{t-j}^2]$$

$$= E(z_t^2 z_{t-1}) + (\phi + \theta)^2 \phi \sum_{j=2}^{\infty} \phi^{2j-4} \sigma^2$$

$$= (\phi + \theta) E[z_t z_{t-1}] + (\phi + \theta)^2 \phi \sum_{j=2}^{\infty} \phi^{2j-4} \sigma^2$$

$$= \sigma^2 \left[\phi + \theta + \frac{(\phi + \theta)^2 \phi}{1 - \phi^2} \right]$$

Case h=2

$$\text{cov}(x_t, x_{t-2}) = E(x_t x_{t-2}) = \sigma^2$$

$$= \sigma^2 \sum_{j=0}^{\infty} \psi_j \psi_{j-2}$$

$$E(x_t x_{t-2})$$

$$= E \left[(z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} z_{t-j}) \cdot \right.$$

$$\left. (z_{t-2} + (\phi + \theta) \sum_{j=3}^{\infty} \phi^{j-3} z_{t-j}) \right]$$

$$= E \left[(z_t + z_{t-1}(\phi + \theta) + z_{t-2}(\phi + \theta)\phi \right.$$

$$\left. + (\phi + \theta)\phi \sum_{j=3}^{\infty} \phi^{j-3} z_{t-j}) \right]$$

$$\left[z_{t-2} + (\phi + \theta) \sum_{j=3}^{\infty} \phi^{j-3} z_{t-j} \right]$$

$$= (\phi + \theta) \phi E[z_{t-2}^2] + (\phi + \theta)^2 \phi^2$$

$$\sum_{j=3}^{\infty} \phi^{2j-6} E(z_{t-j}^2)$$

$$= \sigma^2 \left((\phi + \theta) \phi + \frac{(\phi + \theta)^2 \phi^2}{1 - \phi^2} \right)$$

We see the pattern and follow that

$$\gamma(h) = \sigma^2 \left[1 + \frac{(\phi + \theta)^2 \phi^2}{1 - \phi^2} \right] \text{ if } h=0$$

$$\gamma(h) = \sigma^2 \left[(\phi + \theta) \phi^{h-1} + \frac{(\phi + \theta)^2 \phi^h}{1 - \phi^2} \right] \text{ if } h > 0$$

Estimation of μ :

The moment estimator of the mean μ of a stationary process is the sample mean,

$$\bar{x}_m = m^{-1} (x_1 + x_2 + \dots + x_m)$$

It is an unbiased estimator of μ , since

$$E(\bar{x}_m) = m^{-1} (E x_1 + \dots + E x_m) = \mu.$$

The mean squared error of $\bar{x}_m$ is,

$$E(\bar{x}_m - \mu)^2 = \text{var}(\bar{x}_m)$$

$$= \text{var} [m^{-1} (x_1 + x_2 + \dots + x_m)]$$

$$= m^{-2} \text{var} [x_1 + x_2 + \dots + x_m]$$

$$= m^{-2} [\text{var}(x_1) + \dots + \text{var}(x_m)$$

$$+ \sum_{i \neq j} \text{cov}(x_i, x_j)]$$

$$= m^{-2} \left[\sum_{i=1}^m \sum_{j=1}^m \text{cov}(x_i, x_j) \right]$$

$$\text{cov}(x_i, x_j)$$

$$\text{lag } h = i - j$$

$$\gamma(0) \text{ repeat } m \text{ times}$$

$$\gamma(-1) \text{ repeat } (m-1) \text{ times}$$

$$\gamma(1) \text{ " } (m-1) \text{ "}$$

$$\gamma(2) \text{ " } (m-2) \text{ "}$$

$$\gamma(-2) \text{ " } (m-2) \text{ "}$$

$$\begin{aligned}
 \text{var}[x_m] &= m^{-2} [m r(0) + (m-1) r(-1) + (m-1) r(1) \\
 &\quad + (m-2) r(-2) + (m-2) r(2) + \dots + \\
 &\quad r\{- (m-1)\} + r\{(m-1)\}] \\
 &= m^{-2} [r(0) + \left(\frac{m-1}{m}\right) r(-1) + \left(\frac{m-1}{m}\right) r(1) \\
 &\quad + \left(\frac{m-2}{m}\right) r(-2) + \left(\frac{m-2}{m}\right) r(2) + \dots + \\
 &\quad \frac{1}{m} r\{- (m-1)\} + \frac{1}{m} r\{(m-1)\}]
 \end{aligned}$$

Note that,

$$\begin{aligned}
 \frac{(n-1)}{n} &= 1 - \frac{1}{n} \\
 \left(\frac{n-2}{n}\right) &= 1 - \frac{2}{n} \\
 &\dots \\
 \frac{1}{n} &= 1 - \frac{n-1}{n}
 \end{aligned}$$

$$\begin{aligned}
 &= m^{-1} [r(0) + \left(1 - \frac{1}{m}\right) r(-1) + \\
 &\quad \left(1 - \frac{1}{m}\right) r(1) + \left(1 - \frac{2}{m}\right) r(-2) + \\
 &\quad \left(1 - \frac{2}{m}\right) r(2) + \dots + \left(1 - \frac{n-1}{m}\right) r\{(m-1)\}]
 \end{aligned}$$

$$\begin{aligned}
 &= m^{-1} \sum_{h=-m}^{m-1} \left(1 - \frac{|h|}{m}\right) r(h) \\
 &= m^{-1} \sum_{h=-m}^m \left(1 - \frac{|h|}{m}\right) r(h)
 \end{aligned}$$

Estimation of $r(\cdot)$ and $s(\cdot)$

Suppose the time series is observed at m points.

$$x_1, x_2, x_3, x_4, \dots, x_{m-2}, x_{m-1}, x_m$$

Stationarity implies that

$$\text{cov}(x_1, x_2) = \text{cov}(x_2, x_3) = \dots = \text{cov}(x_{m-2}, x_{m-1})$$

$$\left. \begin{aligned}
 &x_1, x_3 \\
 &x_2, x_4 \\
 &x_3, x_5 \\
 &\vdots \\
 &x_{m-2}, x_m
 \end{aligned} \right\}$$

calculate sample covariance based on these points

The sample autocorrelation function is defined as,

$$\begin{aligned}
 \hat{r}(h) &= \frac{1}{m-h} \sum_{t=1}^{m-h} (x_{t+h} - \bar{x})(x_t - \bar{x}) \\
 &= \frac{1}{m-h} \sum_{t=h+1}^m (x_t - \bar{x})(x_{t-h} - \bar{x})
 \end{aligned}$$

$$\hat{\gamma}(h) = \frac{1}{n} \sum_{i=1}^{n-h} (x_{t+h} - \bar{x})(x_t - \bar{x})$$

$$= \frac{1}{n} \sum_{i=h+1}^n (x_t - \bar{x})(x_{t-h} - \bar{x})$$

We can divide by n instead of $n-h$ as it gives us a positive definite estimator

with $\hat{\gamma}(h) = \hat{\gamma}(-h)$ for $h = 0, 1, 2, \dots, n-1$

The sample autocorrelation function

$$\hat{\rho}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)}$$

• AR(1) $x_t - \phi x_{t-1} = z_t$

$$\Rightarrow x_t - \mu = \phi(x_{t-1} - \mu) + z_t$$

$$|\phi| < 1$$

and $\{z_t\} \sim WN(0, \sigma^2)$

$$\gamma(h) = \frac{\phi^{|h|} \sigma^2}{1 - \phi^2}$$

- positive definite - eigen value matrix

- AC is distn commo; ;

$\Rightarrow$ AC is commo; for inference process
for facilitate work of AC well

interactable (property exchange and)

simple or complex situation

Advantage: For some condition, distn is well approximated by the normal

distn.

$$\hat{\gamma}_t = (\hat{\gamma}(1), \dots, \hat{\gamma}(k))'$$

$$\hat{\gamma} \sim N(\gamma, n^{-1}W)$$

n = sample size
 W is given by the variance-covariance

AR(1)

$$x_t - \phi x_{t-1} = z_t$$

$$\Rightarrow x_t = \phi x_{t-1} + z_t$$

$$\text{var}(x_t) = \text{var}[\phi x_{t-1} + z_t]$$

$$= \phi^2 \text{var}(x_{t-1}) + \sigma^2$$

Recall the x_t is stationary,

so $\text{var}(x_t) = \text{var}(x_{t-1}) = \gamma(0)$

$$\gamma(0) = \phi^2 \gamma(0) + \sigma^2$$

$$\Rightarrow \gamma(0) = \frac{\sigma^2}{1 - \phi^2}$$

$$\text{cov}(x_{t-h}, x_t) = \gamma(h)$$

$$x_t = \phi x_{t-1} + z_t$$

Multiply on both sides by x_{t-h} and take expectations

$$E(x_t x_{t-h}) = \phi E(x_{t-1} x_{t-h}) + E[z_t x_{t-h}]$$

$$\Rightarrow \gamma(h) = \phi \gamma(h-1), \quad h = 1, 2, \dots$$

Let, $h=1$

$$\gamma(1) = \phi \gamma(0) = \phi \frac{\sigma^2}{1-\phi^2}$$

$$\gamma(2) = \phi \gamma(1) = \phi^2 \frac{\sigma^2}{1-\phi^2}$$

$$\vdots$$

$$\gamma(h) = \frac{\phi^{h-1} \sigma^2}{1-\phi^2}$$

$$\gamma(h) = \frac{\gamma(h)}{\gamma(0)} = \frac{\phi^{h-1} \sigma^2}{\sigma^2} = \phi^{h-1}$$

- Since $|\phi| < 1$, the ACF decreases exponentially as the number of lags increases

- If $0 < \phi < 1$, all the correlations are positive

- If $-1 < \phi < 0$, $\gamma(1) < 0$, $\gamma(2) > 0$, $\gamma(3) < 0$ — The signs of all correlations alternate.

MA(1)

$$x_t = \theta(1) z_t$$

$$\text{MA}(1), \quad \gamma(h) = \frac{\sum_{j=0}^{n-h} \theta_j \theta_{j+h}}{1 + \theta_1^2 + \theta_2^2 + \dots + \theta_n^2} = 0, \quad h > 1$$

Need for PACF:

- Cutting off of $\gamma(h)$ after n lags is the signature of MA(1) process.
- If the process is ARMA on AR, the ACF alone tells us little about the order of dependence — orders of dependence
- We want a function that behave like the ACF of MA models, but for AR models.
- PACF

- Least square method is an unbiased, best parameter estimation concept in GLM model, a variance of the estimator.

How to estimate?

MLE (Maximum likelihood function)

Gaussian and Normal 2nd order difference

- Gaussian physics 2 merger,
- similar to the normal distn but not exactly normal,

2591

over good news
error sum of squares

Endo- term,

$$E_{\text{meas}} = \langle X_T - X_T(\hat{\phi}, \hat{\theta}) \rangle_{\mathcal{Z}_T}$$

$$\begin{array}{c} \tau > \\ \tau \\ || \\ \Sigma > \\ + \\ / \\ Q > \end{array}$$

recade $\Rightarrow$

rescale prior $\Rightarrow$ rescale value
rescale value $\Rightarrow$ plot prior distribution
prior against prior for common distribution

=) Glühöfen für Feuerholz
mit

⇒ for informal process: (reschedule)
order owner may remove it.
⇒ Ask user again later!

- Forces thing

- Forces thing

ARMA is more order of
and order of MA process!
order selection of arma , set of p
and q ratio, giving up AIC as value
for model select which.

- FPE and RICE (common)

- SARIMA models:

- 3 means seasonality.
- ARIMA 3 some situation 2 seasonality
- can smooth cover over the seasonality
- For handling ARIMA to SARIMA
- we modified ARIMA only application
- univariate data 2, 3 trend pattern
- also seasonality 3 trend pattern
- can smooth handle both seasonality
- ARIMA model 2, 3

- $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$
 $\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p$
 $\mathbb{I} + \text{adds three more}$
 $\phi(B) \phi(B^5) Y_t = \phi(B) \otimes$

next parameter modify

Hyperparameter Most parameter exist
- no, can't be changed manually
- in 2nd, best hyperparameter
- 3rd, 1st

- put model up earlier completely irrelevant way to, early so earlier well estimate until
- ARIMA generated seasonality from the error term. (No early pattern no seasonality)
- $\rho = \text{period}$
(number of the ARIMA model on period)

- Memory of time series
- Autocorrelation

- Moment of time
- Data set & order correlation
- Data correlated through first and result matrix and sort
- Auto correlation matrix

- correlation for long-term dependencies
- time to time series patterns
- longer, even for memory of the series
- short and long memory (common high)
- prediction easier
- Pearson correlation may vary

TS-13 (Spectral Analysis)

- Time series analysis of series over time

- Location, Time series are indexed by time. Researcher for location best time for adjustment or series for plot variable, location series for continuous movement (order com in series 3, 2, 1) (order com in series 3, 2, 1)

[illegible]

- Example - series for AR(1) exist for $\rho < 1$, $\{x_t\}$ stationary process $2\pi \leq \omega < 2\pi + 2\pi$

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\omega h} \gamma(h)$$

$\{x_t\}$ is a function of $f(\omega)$ and $\sqrt{-1}$
 $e^{i\omega} = \cos(\omega) + i\sin(\omega)$ and $i = \sqrt{-1}$

Property: $|e^{-i\omega h}|^2 = \cos^2(\omega h) + \sin^2(\omega h) = 1$

- converge over observation for AR(1)
- $f(\omega)$ must be symmetric
- Its nonnegative

$$\gamma(h) = \begin{cases} 1 & \text{if } h=0 \\ -0.5 & \text{if } h=\pm 1 \\ -0.25 & \text{if } h=\pm 2 \\ 0 & \text{if } h=\pm 3 \end{cases}$$

Since $\gamma(h) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\omega h} \gamma(h)$

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\omega h} \gamma(h) \\ &= \frac{1}{2\pi} [1 - 0.5(e^{-2i\omega} + e^{2i\omega}) - 0.25(e^{-4i\omega} + e^{4i\omega})] \\ &= \frac{1}{2\pi} [1 - \cos 2\omega - 0.5 \cos 4\omega] \end{aligned}$$

$\gamma(h)$ can not be an autocovariance function.

- spectral density of an MA(1) process

$$x_t = z_t + \theta z_{t-1}$$

where $\{z_t\} \sim WN(0, \sigma^2)$, then

from u.1.1

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{2\pi} [1 + \theta^2 + 2\theta \cos \omega] \\ &= \frac{\sigma^2}{2\pi} [1 + 2\theta \cos \omega + \theta^2] \end{aligned}$$

- spectral density of an AR(1) process

$$x_t = \phi x_{t-1} + z_t$$

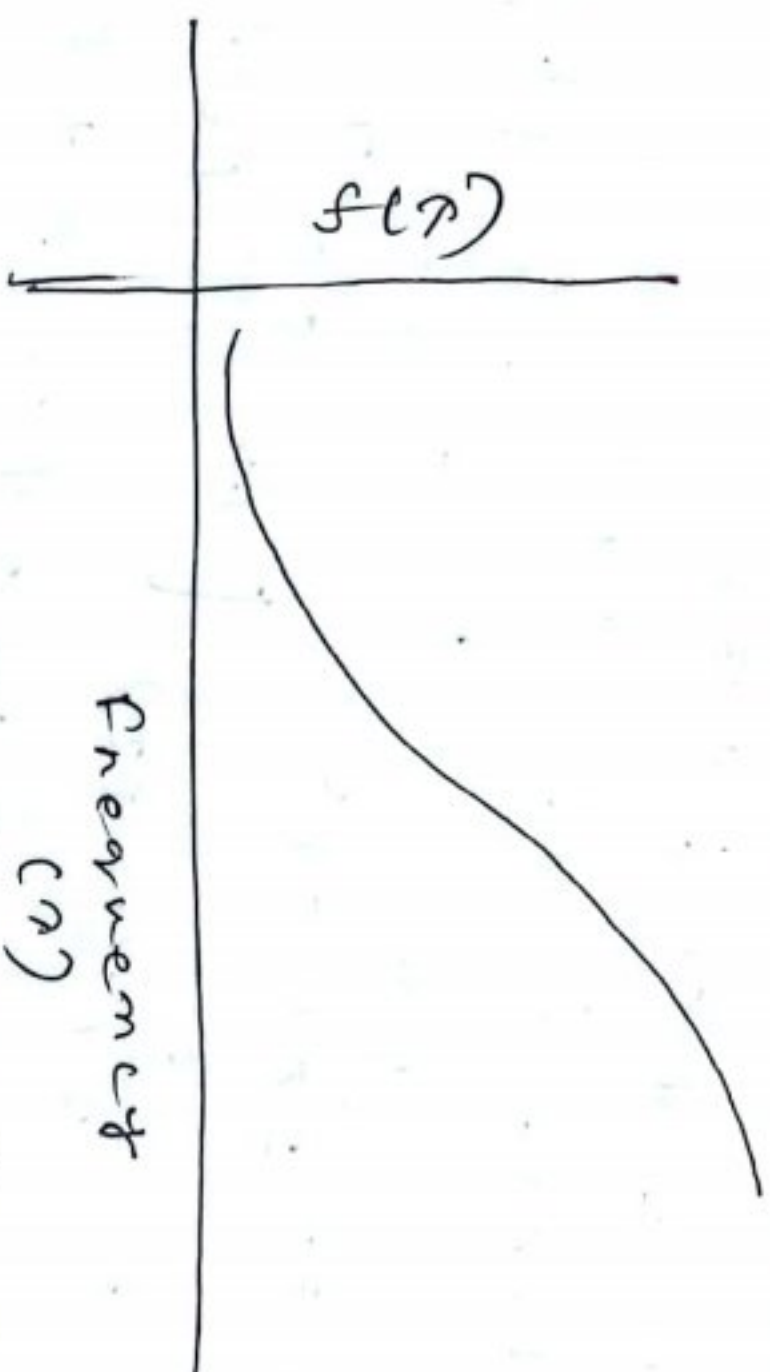
where $\{z_t\} \sim WN(0, \sigma^2)$

$$\begin{aligned} f(\omega) &= \frac{\sigma^2}{2\pi(1-\phi^2)} \left[1 + \sum_{h=1}^{\infty} \phi^h (e^{-i\omega h} + e^{i\omega h}) \right] \\ &= \frac{\sigma^2}{2\pi(1-\phi^2)} \left[1 + \frac{\phi e^{i\omega}}{1-\phi e^{i\omega}} + \frac{\phi e^{-i\omega}}{1-\phi e^{-i\omega}} \right] \\ &= \frac{\sigma^2}{2\pi} [1 - 2\phi \cos \omega + \phi^2]^{-1} \end{aligned}$$

$$X_t = Z_t - 0.4Z_{t-1} \sim WN(0, 1.02)$$

$$\begin{aligned} f(\lambda) &= \frac{0.2}{2\pi} [1 + 2 \cos \lambda + 0.2] \\ &= \frac{0.2}{2\pi} [1 + 2 \times (-0.4) \cos \lambda + (-0.4)^2] \\ &= \frac{1}{2\pi} [1 - 0.8 \cos \lambda + 0.16] \\ &= \frac{1}{2\pi} [1 - 0.8 \cos \lambda + 0.16] \end{aligned}$$

TS (13) Spectral analysis



The spectral density $f(\lambda)$, $0 \leq \lambda \leq \pi$
of $X_t = Z_t - 0.4Z_{t-1}$, where
 $\{Z_t\} \sim WN(0, 0.2)$



The spectral density of $f(\lambda)$, $0 \leq \lambda \leq \pi$
of $X_t = Z_t + 0.4Z_{t-1}$ where
 $\{Z_t\} \sim WN(0, 0.2)$

• $\{X_t\}$ time series
Aver. $2\pi m f(\lambda) \rightarrow h \log$

According

Spectral density AR(1)

AR(1) process:

$$X_t = \phi X_{t-1} + Z_t \quad ; Z_t \sim WN(0, \sigma^2)$$

$$\gamma(h) = \begin{cases} \frac{\sigma^2}{1-\phi^2} & ; h=0 \\ \phi^{|h|} \gamma(0) & ; |h| > 0 \end{cases}$$

Now, the spectral density of X_t ,

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\omega h} |\gamma(h)| \quad ; -\infty < \omega < \infty$$

$$= \frac{1}{2\pi} \left[\gamma(0) + \sum_{h=1}^{\infty} (e^{-i\omega h} + e^{i\omega h}) \phi^h \gamma(0) \right]$$

$$= \frac{1}{2\pi} \left[\gamma(0) + \sum_{h=1}^{\infty} (e^{-i\omega h} + e^{i\omega h}) \phi^h \right]$$

$$= \frac{\gamma(0)}{2\pi} \left[1 + \sum_{h=1}^{\infty} (\phi e^{-i\omega} + \phi e^{i\omega})^h \right]$$

$$= \frac{\gamma(0)}{2\pi} \left[1 + \sum_{h=1}^{\infty} (\phi e^{-i\omega} + \phi e^{i\omega})^h \right]$$

$$= \frac{\gamma(0)}{2\pi} \left[1 + \frac{\phi e^{-i\omega}}{1 - \phi e^{-i\omega}} + \frac{\phi e^{i\omega}}{1 - \phi e^{i\omega}} \right]$$

$$= \frac{\sigma^2}{2\pi(1-\phi^2)} \cdot \frac{(1-\phi e^{-i\omega})(1-\phi e^{i\omega})}{(1-\phi e^{-i\omega})(1-\phi e^{i\omega}) + \phi e^{-i\omega}(1-\phi e^{i\omega}) + \phi e^{i\omega}(1-\phi e^{-i\omega})}$$

$$= \frac{\sigma^2}{2\pi(1-\phi^2)} \left[\frac{1-\phi^2}{1-\phi(e^{-i\omega}+e^{i\omega})+\phi^2} \right]$$

$$= \frac{\sigma^2}{2\pi} \left[\frac{1-2\phi \cos \omega + \phi^2}{1-\phi(e^{-i\omega}+e^{i\omega})+\phi^2} \right]$$

Spectral density MA(1)

MA(1) process: $X_t = Z_t + \theta Z_{t-1}$

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & ; h=0 \\ \theta \sigma^2 & ; h=\pm 1 \\ 0 & ; |h| > 1 \end{cases}$$

$$\gamma(h) = \begin{cases} \sigma^2(1+\theta^2) & ; h=0 \\ \theta \sigma^2 & ; h=\pm 1 \\ 0 & ; |h| > 1 \end{cases}$$

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\omega h} \gamma(h)$$

$$= \frac{1}{2\pi} \left[\gamma(0) + e^{-i\omega} \gamma(1) + e^{i\omega} \gamma(-1) + 0 \right]$$

$$= \frac{1}{2\pi} \left[\sigma^2(1+\theta^2) + \theta \sigma^2 [e^{-i\omega} + e^{i\omega}] \right]$$

$$= \frac{\sigma^2}{2\pi} [1 + \theta^2 + \theta(e^{-i\pi} + e^{i\pi})]$$

$$= \frac{\sigma^2}{2\pi} [1 + 2\theta \cos \pi + \theta^2]$$

ARMA (1,1)

$$X_t - \phi X_{t-1} = Z_t + \theta Z_{t-1}$$

$$\gamma(h) = \begin{cases} \sigma^2 \left[1 + \frac{(\phi + \theta)^2}{1 - \phi^2} \right] & ; h = 0 \\ \sigma^2 \left[(\phi + \theta) \cdot \rho_{h-1} + \frac{(\phi + \theta)^2 \phi^h}{1 - \phi^2} \right] & ; |h| > 0 \end{cases}$$

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ihn} \gamma(h)$$

$$= \frac{1}{2\pi} \left[\gamma(0) + \sum_{h=1}^{\infty} (e^{-ihn} + e^{ihn}) \gamma(h) \right]$$

$$= \frac{1}{2\pi} \left[\frac{(1 + \theta^2 + 2\theta \cos \pi)}{1 + \phi^2 - 2\phi \cos \pi} \right]$$

$$\phi(\beta) X_t = \theta(\beta) Z_t$$

$$f(\omega) = \frac{\sigma^2}{2\pi} \frac{1 + \theta(e^{i\pi})}{1 + \phi(e^{i\pi})}$$

$\{X_t\} \rightarrow$ Time series
 $\gamma(h) \rightarrow$ ACF ; $h = \log$
 • with various other function and $\gamma(h)$
 $\gamma(h)$ from $[-\pi, \pi]$ with $\rho(-\pi) = 0$
 giving upper according to Fourier transformation, see $\gamma(h)$ for ACF function transformation, see $\gamma(h)$ for ACF function transformation

$F =$ cumulative type function for all integers

$$\gamma(h) = \int_{(-\pi, \pi)} e^{ihn} dF(\omega)$$

$\downarrow$ spectral distn function $\gamma(h)$ discrete
 $F = \begin{cases} \text{discrete distn} & \text{--- } f(\omega) \text{ discrete} \\ \text{continuous distn} & \text{--- } f(\omega) \text{ continuous} \end{cases}$

AR(1)

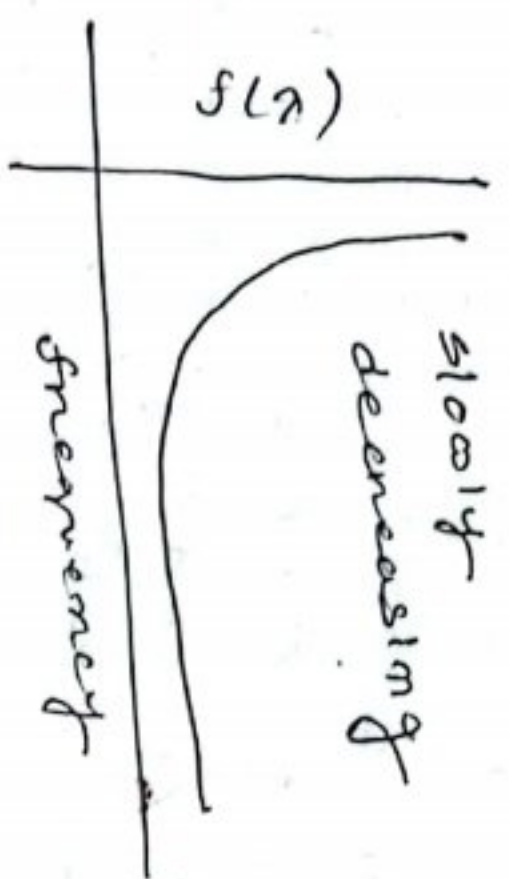
$$X_t = \phi X_{t-1} + Z_t$$

$$\therefore \gamma(h) = \begin{cases} \frac{\sigma^2}{1 - \phi^2} & ; h = 0 \\ \phi^h \gamma(0) & ; |h| > 0 \end{cases}$$

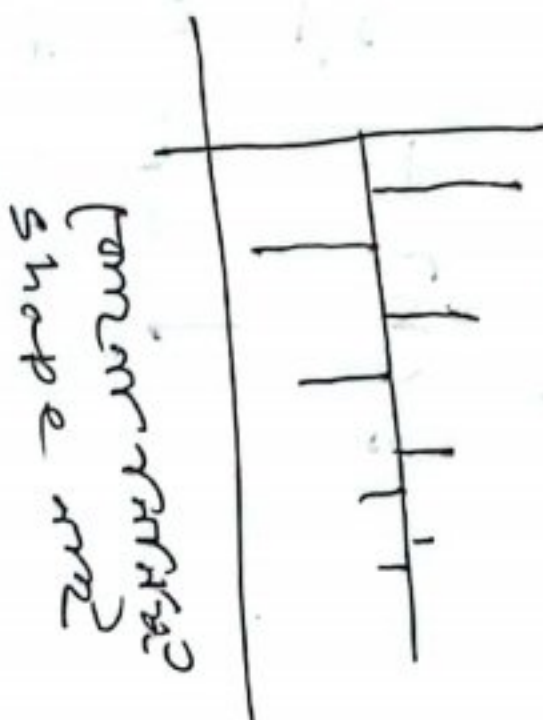
$$\therefore f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ihn} \gamma(h)$$

$$= \frac{\sigma^2}{2\pi} (1 - 2\phi \cos \pi + \phi^2)$$

ϕ = positive $2\pi n$



ϕ = negative $2\pi n$



unit root case and non-stationarity process

Periodogram:

$\text{Im}(\cdot)$ is a analogue of $2\pi f(\cdot)$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{C}^m \rightarrow$ vector of complex numbers

Vector Representation - direction vector

$\omega_k = \frac{2\pi k}{m}$; k is any integer

$\frac{m-1}{2} \leq k \leq \frac{m}{2}$

• FM - Fourier frequencies associated with sample size m .

↓
A subset of $[-\pi, \pi]$

$a_k = \frac{1}{\sqrt{m}} \begin{bmatrix} e^{i\omega_k} \\ e^{i2\omega_k} \\ \vdots \\ e^{im\omega_k} \end{bmatrix}$

orthogonal basis and matrix
system with identity matrix

$a_j^* a_k = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$

Most energy from transformation and best fit for transformation of time series components depend on how and time invariant, combination of signals and time invariant

3rd tutorial

Previous questions:

- i) Define an ARMA(p,q) process. When it is called ci) invertible and ii) non-invertible process?

e) Check whether or not the following process is univ. causal and invertible, where $z_t \sim WN(0, \sigma^2)$

- (i) $X_t + 0.2X_{t-1} - 0.48X_{t-2} = z_t$
- (ii) $X_t + 0.6X_{t-1} = z_t + 1.2z_{t-1}$
- (iii) $X_t + 1.8X_{t-1} + 0.81X_{t-2} = z_t$

3)

MA(1) process

$$\gamma(h) = \begin{cases} (1+\theta^2)\sigma^2 & ; \text{for } h=0 \\ \theta\sigma^2 & ; \text{for } h=\pm 1 \\ 0 & ; \text{for } |h| > 1 \end{cases}$$

$$\gamma(h) = \begin{cases} 1 & ; \text{for } h=0 \\ \frac{\theta}{1+\theta^2} & ; \text{for } h=\pm 1 \\ 0 & ; \text{for } |h| > 1 \end{cases}$$

MA(2) process:

$$\gamma(h) = \begin{cases} (1+\theta_1^2+\theta_2^2)\sigma^2 & ; \text{for } h=0 \\ (\theta_1+\theta_1\theta_2)\sigma^2 & ; \text{for } h=\pm 1 \\ \theta_2\sigma^2 & ; \text{for } |h|=2 \\ 0 & ; \text{for } |h| > 2 \end{cases}$$

$$\gamma(h) = \begin{cases} \frac{\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & ; \text{for } h=0 \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & ; \text{for } h=\pm 1 \\ 0 & ; \text{for } |h| > 2 \end{cases}$$

$$\gamma(h) = \begin{cases} \frac{\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & ; \text{for } h=0 \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & ; \text{for } h=\pm 1 \\ 0 & ; \text{for } |h| > 2 \end{cases}$$

T5-15 (Multivariate time series)

- $X_t = (x_{t1}, x_{t2})'$
Bivariate time series
more than two variable present
multivariate.
regress variable (or column) against
rest judge error-ss,
Time fixed $\rightarrow$ variable add error

- Mean vector,
$$\mu_t = E X_t = \begin{bmatrix} E x_{t1} \\ E x_{t2} \end{bmatrix} = \begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \end{bmatrix}$$

covariance matrices,

$$\begin{aligned} \Gamma(t+h, t) &= \text{cov}(x_{t+h}, x_t) \\ &= \begin{bmatrix} \text{cov}(x_{t+h,1}, x_{t1}) & \text{cov}(x_{t+h,1}, x_{t2}) \\ \text{cov}(x_{t+h,2}, x_{t1}) & \text{cov}(x_{t+h,2}, x_{t2}) \end{bmatrix} \end{aligned}$$

$$\begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}_{2 \times 2}$$

$$\begin{aligned} \Gamma(h) &= \text{cov}(x_{t+h}, x_t) \\ &= \begin{bmatrix} \gamma_{11}(h) & \gamma_{12}(h) \\ \gamma_{21}(h) & \gamma_{22}(h) \end{bmatrix} \end{aligned}$$

- μ and covariance matrix lagged
error not zero, not on the time.
- mean and variance approximately
constant over stationary but error
correlation error judge error-ss

- weakly stationary

- (i) μ_t is independent of t
- (ii) $\Gamma(x_{t+h}, t)$ is independent of t for each h .

- $R(h) = \begin{bmatrix} \gamma_{11}(h) & \dots & \gamma_{1m}(h) \\ \vdots & \ddots & \vdots \\ \gamma_{m1}(h) & \dots & \gamma_{mm}(h) \end{bmatrix}$

It is a symmetric matrix

- 2nd order property - mean and
variance should be finite.

- Maximum likelihood estimation method

$$\begin{aligned} \bar{X}_m &= \frac{1}{m} \sum_{t=1}^m x_t \\ \hat{\Gamma}(h) &= \begin{cases} \frac{1}{m-1} \sum_{t=1}^{m-h} (x_{t+h} - \bar{X}_m)(x_t - \bar{X}_m)' & \text{for } 0 \leq h \leq m \\ \hat{\Gamma}(-h)' & \text{for } -m+1 \leq h < 0 \end{cases} \end{aligned}$$

$$\hat{\Sigma}_{ij}(h) = \hat{\gamma}_{ij}(h) \{ \hat{\gamma}_{ii}(0) \hat{\gamma}_{jj}(0) \}^{-1/2}$$

$$\mu_t = E X_t = \begin{bmatrix} \mu_{t+1} \\ \vdots \\ \mu_{t+m} \end{bmatrix}$$

- white noise $\{z_t\} \sim WN(0, \Sigma)$
IID noise $\{z_t\} \sim iid(0, \Sigma)$

- ARMA original series or white noise plus mean linear combination errors
system over time, over linear process

ARMA

Backshift operator:

The backshift operation, B is a useful notational device when working with time series lags.

$$B X_t = X_{t-1}$$

$$B(B X_t) = B^2 X_t = X_{t-2}$$

$$y_t' = y_t - y_{t-1} = (1-B)y_t$$

$$y_t'' = y_t - 2y_{t-1} + y_{t-2} = (1-B)^2 y_t$$

Distribution of ACF:

- For inference we need sampling distn of ACF.
- Although the distn of $\hat{g}(h)$ is intractable for samples from even the simplest time series models, it can usually be well approximated by a normal distribution for large sample sizes.

Example:

For ARMA models,

$\hat{g}_k = (\hat{g}(1), \dots, \hat{g}(k))'$ is approximately distributed for large n as

$$N(g_k, m^{-1}W) \text{ i.e.,}$$

$$\hat{g} \sim N(g, m^{-1}W)$$

W is the covariance matrix whose (i, j) element is given by Bartlett's formula

$$w_{ij} = \sum_{k=1}^{\infty} \{ g(k+i) + g(k-i) - 2g(j)g(k) \} \times \{ g(k+j) + g(k-j) - 2g(i)g(k) \}$$

DAY-1

TS-11 and TS-15

1) Explain the multivariate time series with example. Also, discuss the second order properties of a multivariate time series.

2) How does a univariate time series differ from the multivariate time series? Explain with an example. Define a linear process for a multivariate time series.

Solution:

(1) Multivariate time series is a series of m -dimensional vectors $\{X_t\}$ = $(X_{t1}, X_{t2}, \dots, X_{tm})'$ observed at time t , where $t = 1, 2, \dots$ and it is assumed that the m component series are dependent.

Let's say we want to analyse the daily stock market performance of a company. Instead of looking at just one variable, such as the stock's closing price, we gather several variables over time to create a

multivariate time series dataset. These

Date	closing price	Volume	Market Index
2023.10.01	106.50	1,000,000	4500
2023.10.02	101.20	950,000	4510
2023.10.03	102.10	1,200,000	4525
...	...	...	...

Analyzing this multivariate time series can help us understand how these variables are related and how they impact the company's stock price.

Second order properties:

The second order properties of the multivariate time series $\{X_t\}$ are specified by the mean vectors

$$\mu_t = E X_t = \begin{bmatrix} \mu_{t1} \\ \vdots \\ \mu_{tm} \end{bmatrix}$$

and covariance matrices

$$\Gamma(t+h, t) = \begin{bmatrix} \gamma_{11}(t+h, t) & \dots & \gamma_{1m}(t+h, t) \\ \vdots & & \vdots \\ \gamma_{m1}(t+h, t) & \dots & \gamma_{mm}(t+h, t) \end{bmatrix}$$

where, $\gamma_{ij}(t+h, t) = \text{cov}(X_{t+h, i}, X_{t, j})$

(2) Key Differences

Aspect	Univariate time series	Multivariate time series
Number of variables	One variable at each time point.	Multiple variables at each time point.
Data interpretation	Focused on a single variable behavior.	Considers interaction between variables.
Complexity of analysis	Simpler	More complex due to multiple variables and dependencies.

Univariate time series

A univariate time series consists of a single time-dependent variable recorded over a sequence of equally spaced time intervals.

- The daily temperature of a city over a year.

Date	Temperature (°C)
2023.09.01	10.2
2023.09.02	9.8
2023.09.03	11.5
...	...

A multivariate time series, on the other hand, consists of multiple time-dependent variables recorded over a sequence of equally spaced time intervals.

Linear process:

The multivariate series $\{x_t\}$ is a linear process if it has the representation

$$x_t = \sum_{j=-\infty}^{\infty} c_j z_{t-j}, \quad \{z_t\} \sim WN(0, I)$$

where $\{c_j\}$ is a sequence of $m \times m$ matrices whose components are absolutely summable.

T5-11 (Related previous year questions)

(1) Define an ARMA(p,q) process.

When it is called (i) invertible and

(ii) non-invertible process?

Soln:

$\{X_t\}$ is an ARMA(p,q) process if

$\{X_t\}$ is stationary and if for every t,

$$X_t - \phi_1 X_{t-1} - \dots - \phi_p X_{t-p}$$

$$= Z_t + \theta_1 Z_{t-1} + \dots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and the polynomials $(1 - \phi_1 z - \dots - \phi_p z^p)$ and

$(1 + \theta_1 z + \dots + \theta_q z^q)$ have no common factors.

Invertibility:

An ARMA(p,q) process $\{X_t\}$ is

invertible if there exist constants

$\{\pi_j\}$ such that $\sum_{j=0}^{\infty} |\pi_j| < \infty$ and

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j} \text{ for all } t.$$

Invertibility is equivalent to the condition

$$\theta(z) = (1 + \theta_1 z + \dots + \theta_q z^q) \neq 0 \text{ for all } |z| \leq 1$$

Note:

Existence and uniqueness:

A stationary solution $\{X_t\}$ of equations

(3.1.1) exists if and only if

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p \neq 0 \text{ for all } |z| = 1$$

Causality:

An ARMA(p,q) process $\{X_t\}$ is causal, on a causal function of $\{Z_t\}$, if there exists constants $\{\psi_j\}$ such that

$$\sum_{j=0}^{\infty} |\psi_j| < \infty \text{ and}$$

$$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} \text{ for all } t.$$

Causality equivalent to the condition

$$\phi(z) = 1 - \phi_1 z - \dots - \phi_p z^p \neq 0 \text{ for all } |z| < 1$$

✓ (c2) check whether or not the following process is unique, causal and invertible, where $\{z_t\} \sim WN(0, \sigma^2)$

(i) $x_t + 0.2x_{t-1} - 0.48x_{t-2} = z_t$

(ii) $x_t + 0.6x_{t-1} = z_t + 1.2z_{t-1}$

(iii) $x_t + 1.8x_{t-1} + 0.81x_{t-2} = z_t$

Soln:

(i) Here the autoregressive polynomial

is, $\phi(z) = 1 + 0.2z - 0.48z^2$

$= (1 + 0.8z)(1 - 0.6z)$

the autoregressive polynomial is zero at

$z = -1/0.8$ or $z = 1/0.6$

since both of these are located outside the unit circle, the process is causal.

Process is obviously invertible

since $z_t = x_t + 0.2x_{t-1} - 0.48x_{t-2}$

[AR(2) process is always invertible]

(ii)

$\phi(z) = 1 + 0.6z$

$\phi(z) = 0$ at $z = -10/6$ (causal)

$\theta(z) = 1 + 1.2z$

$z = -5/6$ (not invertible)

(iii) $\phi(z) = 1 + 1.8z + 0.81z^2$

$= (1 + 0.9z)^2$

$z = -10/9$ (causal)

also invertible.

✓ (c3)

show that $\{x_t\}$, $x_t = \sum_{j=-\infty}^{\infty} \psi_j z_{t-j}$ where $\{z_t\} \sim WN(0, \sigma^2)$ and

$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty$, is a stationary time series with mean 0 and autocovariance

function,

$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2$

Solm₀

$$\begin{aligned} E(X_t) &= E\left(\sum_{j=-\infty}^{\infty} \psi_j z_{t-j}\right) \\ &= \sum_{j=-\infty}^{\infty} \psi_j E[z_{t-j}] \\ &= 0 \quad ; \text{ since } E z_t = 0 \end{aligned}$$

and,

$$\begin{aligned} \gamma_X(h) &= E(X_{t+h}, X_t) \\ &= E\left[\left(\sum_{j=-\infty}^{\infty} \psi_j z_{t+h-j}\right)\left(\sum_{k=-\infty}^{\infty} \psi_k z_{t-k}\right)\right] \end{aligned}$$

$$\begin{aligned} &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k E(z_{t+h-j} z_{t-k}) \quad \dots (i) \\ &= \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \psi_j \psi_k \gamma_z(h-j+k) \end{aligned}$$

Since $\{z_t\}$ is a white noise
then $\gamma_z(h-j+k) = 0$ if $k \neq j+h$
and 0 otherwise.

$$\gamma_X(h) = \sum_{j=-\infty}^{\infty} \psi_j \psi_{j+h} \sigma^2 \quad (\text{shown})$$

✓ Ch Find the mean and autocovariance function of the following process;

(i) $X_t = z_t - 0.7z_{t-1} + 0.5z_{t-3}$

(ii) $X_t = z_t z_{t-2}$

(iii) $X_t = z_t + 0.8z_{t-3}$

where $\{z_t\} \sim WN(0,1)$. Hence,
check the stationarity of the process.

Solm₀

(i) $X_t = z_t - 0.7z_{t-1} + 0.5z_{t-3}$

$$\begin{aligned} E(X_t) &= E[z_t - 0.7z_{t-1} + 0.5z_{t-3}] \\ &= E[z_t] - 0.7E[z_{t-1}] + 0.5E[z_{t-3}] \end{aligned}$$

Since, $E[z_t] = 0$

So, $E[z_{t-1}] = E[z_{t-3}] = 0$

$\therefore E(X_t) = 0$

$E(X_{t+1}) = 0$

$$\gamma(t+h, t) = \gamma(h)$$

$$= \text{cov}(x_{t+h}, x_t)$$

$$= E[x_{t+h}, x_t]$$

since $E(x_t)$

$$= E(x_{t+h})$$

$$= 0$$

When, $h=0$

$$\gamma(0) = E[x_t, x_t]$$

$$= E[x_t^2]$$

$$= E[(z_t - 0.7z_{t-1} + 0.5z_{t-3})^2]$$

$$= E[z_t^2] + (0.7)^2 E[z_{t-1}^2]$$

$$+ (0.5)^2 E[z_{t-3}^2]$$

$$= 1 + (0.7)^2 \times 1 + (0.5)^2 \times 1$$

$$= 1.74$$

$$\gamma(h) = E[x_t, x_{t+h}]$$

$$x_{t+h} = z_{t+h} - 0.7z_{t-1+h} +$$

$$0.5z_{t-3+h}$$

$$= z_{t+h} - 0.7z_{t+h-1} +$$

$$0.5z_{t+h-3}$$

$$\therefore \gamma(h) = E[z_t z_{t+h}] - 0.7 E[z_t z_{t+h-1}]$$

$$+ 0.5 E[z_t z_{t+h-3}] - 0.7 E[z_{t-1} z_{t+h-1}]$$

$$+ (0.7)^2 E[z_{t-1} z_{t+h-1}]$$

$$- (0.5 \times 0.7) E[z_{t-1} z_{t+h-3}]$$

$$+ 0.5 E[z_{t-3} z_{t+h}] - (0.7 \times 0.5)$$

$$E[z_{t-3}, z_{t+h-1}] + (0.5)^2 E[z_{t-3} z_{t+h}]$$

$$\gamma(0) = 1.74$$

$$\gamma(1) = 0 - 0.7 E[z_t^2] - 0.7 \times 1$$

$$= -0.7$$

So, the process is stationary.

(ii)

$$x_t = z_t z_{t-2}$$

$$\text{Mean} = E(x_t) = E[z_t z_{t-2}] = 0$$

$$\text{Acf} \gamma(h) = \text{cov}(x_{t+h}, x_t)$$

$$= E(x_{t+h}, x_t)$$

$$= E[(z_{t+h} z_{t+h-2})(z_t z_{t-2})]$$

$$= E[z_{t+h} z_{t+h-2}] \cdot E[z_t z_{t-2}]$$

$$\text{cov}(x_t, x_{t+h}) = E[z_t \cdot z_{t-h}]$$

$$= E[z_{t+h}, z_{t+h-h}]$$

$$= 0 + 0$$

$$= 0$$

$$r(0) = E[z_t^2 z_{t-2}]$$

$$= E[z_{t-2}] + E[z_{t-2}]$$

$$= 1 + 1 = 2$$

$$\therefore r(h) = \begin{cases} 1 & \text{if } h=0 \\ 0 & \text{if } h \neq 0 \end{cases}$$

$$(ii) \quad x_t = z_t + 0.8 z_{t-3}$$

$$\text{Mean}, E(x_t) = E[z_t + 0.8 z_{t-3}]$$

$$= E[z_t] + 0.8 E[z_{t-3}]$$

$$= 0 + 0.8 \times 0$$

$$= 0$$

$$r(h) = \text{cov}(x_{t+h}, x_t)$$

$$= E[x_{t+h}, x_t]$$

$$= E[z_{t+h} + 0.8 z_{t+h-3}]$$

$$[z_{t+h} + 0.8 z_{t+h-3}]$$

$$= E[z_{t+h} (z_t + 0.8 z_{t-3}) + 0.8 (z_{t+h-3}) (z_t + 0.8 z_{t-3})]$$

$$= E[z_{t+h} z_t + 0.8 z_{t+h} z_{t-3} + 0.8 z_{t+h-3} z_t + (0.8)^2 z_{t+h-3} z_{t-3}]$$

$$= E[z_{t+h} z_t] + 0.8 E[z_{t+h} z_{t-3}] + 0.8 E[z_{t+h-3} z_t] + (0.8)^2$$

$$E[z_{t+h-3} z_{t-3}]$$

$$h=0, \quad r(0) = E[z_t^2] + (0.8)^2 E[z_{t-3}^2]$$

$$= 1 + (0.8)^2 \times 1$$

$$= 1.64$$

$$h=1, \quad r(1) = 0$$

$$h=3, \quad r(3) = 0.8 E[z_{t-3}^2]$$

$$= 0.8 \times 1 = 0.8$$

✓ (5) What is linear process? How to transform a second order stationary process to a linear process?

Soln:

The time series $\{x_t\}$ is a linear process if it has the representation

$$x_t = \sum_{j=-\infty}^{\infty} \psi_j z_{t-j} \quad \text{for all } t,$$

where $\{z_t\} \sim WN(0, \sigma^2)$ and $\{\psi_j\}$ is sequence of constants with

$$\sum_{j=-\infty}^{\infty} |\psi_j| < \infty.$$

• Every second order stationary process is either a linear process or can be transformed to a linear process by subtracting a deterministic component. (Wold's decomposition)

✓ (6) What are the different methods to calculate ACF of a time series? calculate the ACF of the MA(2) process:

Soln:

(i) $x_t = z_t + 0.3z_{t-1} - 1.2z_{t-2}$

(ii) $x_t = z_t + 0.3z_{t-1} - 0.4z_{t-2}$
 $\{z_t\} \sim WN(0, 1)$

(i) $x_t = z_t + 0.3z_{t-1} - 1.2z_{t-2}$

MA(2)

$$\gamma(h) = \begin{cases} (1 + 0.12 + 0.22)\sigma^2 & ; h=0 \\ (0.1 + 0.1022)\sigma^2 & ; h=\pm 1 \\ 0 & ; |h| > 1 \end{cases}$$

$$\begin{aligned} \gamma(0) &= [1 + (0.3)2 + (-1.2)2] \sigma^2 \\ &= 2.53 \times 1 = 2.53 \end{aligned}$$

$$\begin{aligned} \gamma(1) &= [0.3 + (0.3)(-1.2)] \times 1 \\ &= -0.06 \end{aligned}$$

$$\gamma(2) = (-1.2) \times 1 = -1.2$$

$$r(h) = \begin{cases} 2.53 & \text{for } h=0 \\ -0.06 & \text{for } h=\pm 1 \\ -1.2 & \text{for } h=\pm 2 \\ 0 & \text{for } |h| > 2 \end{cases}$$

$$g(h) = \begin{cases} 1 & \text{for } h=0 \\ -0.02 & \text{for } h=\pm 1 \\ -0.47 & \text{for } h=\pm 2 \\ 0 & \text{for } |h| > 2 \end{cases}$$

(ii) $X_t = Z_t + 0.3Z_{t-1} - 0.4Z_{t-2}$

$$\begin{aligned} r(0) &= (1 + 0.12 + 0.24) \sigma^2 \\ &= 1 + (0.3)^2 + (-0.4)^2 \\ &= 1.25 \end{aligned}$$

$$\begin{aligned} r(1) &= (0.1 + 0.102) \sigma^2 \\ &= 0.3 + (0.3) \times (-0.4) \\ &= 0.18 \end{aligned}$$

$$r(2) = 0.2 \sigma^2 = -0.4$$

$$\therefore r(h) = \begin{cases} 1.25 & \text{for } h=0 \\ 0.18 & \text{for } h=\pm 1 \\ -0.4 & \text{for } h=\pm 2 \\ 0 & \text{for } |h| > 2 \end{cases}$$

$$g(h) = \begin{cases} 1 & \text{for } h=0 \\ 0.14 & \text{for } h=\pm 1 \\ -0.32 & \text{for } h=\pm 2 \\ 0 & \text{for } |h| > 2 \end{cases}$$

Methods of calculating ACF:

- Direct method
- Sample autocovariance function
- ACF plot
- Yule Walker equations
- Fast Fourier Transform (FFT)
- Parametric modeling
- Cross covariance function

Note:

MA(1)

$$r(h) = \begin{cases} (1 + \theta^2) \sigma^2 & ; h=0 \\ \theta \sigma^2 & ; h=\pm 1 \\ 0 & ; |h| > 1 \end{cases}$$

✓ (7) Express an ARMA (p,q) process in terms of backward shift operator. How could we select the order of an ARMA process?

Soln:

$$\phi(B)X_t = \theta(B)Z_t, \quad Z_t \sim WN(0, \sigma^2)$$

B is the backward shift operator.

Selecting the order of an ARMA process:

Use information criteria like the

Akaike information criterion (AIC)

or the Bayesian information criterion (BIC) to compare different ARMA models with different orders.

ARMA model with various

- Fit ARMA models with various combinations of p and q.
- Calculate the AIC or BIC for each model.
- Select the model with lowest AIC or BIC as the best fitting model.

• After selecting an order, perform diagnostic checks on the residuals of the fitted ARMA model to ensure that the model captures the underlying pattern in the data.

✓ (8) Determine whether the following ARMA process is causal and/or invertible.

$$(i) X_t - 1.9X_{t-2} = Z_t + 0.8Z_{t-1} + 0.7Z_{t-2}$$

$$(ii) X_t + 1.3X_{t-2} = Z_t + 0.3Z_{t-1} - 1.2Z_{t-2}$$

where $Z_t \sim WN(0, 1)$

Soln:

$$\phi(z) = 1 - 1.9z^2$$

$$z = +0.725 \text{ or } z = -0.725$$

It is not causal.

$$\theta(z) = 1 + 0.8z + 0.7z^2$$

$$z = \frac{-0.8 \pm \sqrt{(0.8)^2 - 4(1)(0.7)}}{2(0.7)} = \frac{-0.8 \pm \sqrt{-2.16}}{1.4} = \frac{-0.8 \pm \sqrt{2.16}i}{1.4}$$

$$= \frac{-0.8 \pm 1.42i}{1.4} = \frac{-0.8 \pm i\sqrt{2.16}}{1.4}$$

$$\text{and } |z|^2 = [(-0.8)^2 + 2.16] / (1.4)^2 = 1.429 > 1$$

which implies the process is invertible

(ii)

$$X_t + 1.3X_{t-2} = Z_t + 0.3Z_{t-1} - 1.2Z_{t-2}$$

$$\phi(z) = 1 + 1.3z^2$$

$$z = 0.877 \text{ on } z = -0.877$$

So, the process is not causal

$$\theta(z) = 1 + 0.3z - 1.2z^2$$

$$z = 1.046 \text{ on } z = -0.79$$

Not invertible.

✓ (e) What do you mean by ARCP and MA(q) processes. State the condition under which ARMA(1,1) has unique stationary solution and hence find it.

Soln:

ARCP? Which means that the process is autoregressive process of order p.

Let, $\{X_t\}$ is said to be ARCP process if

$$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \dots + \phi_p X_{t-p} + Z_t$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\phi_1, \phi_2, \dots, \phi_p$ are constants.

MA(q)? Which means that the process is moving average process of order q.

Let, $\{X_t\}$ is said to be a MA(q) process if

$$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \dots + \theta_q Z_{t-q}$$

where $\{Z_t\} \sim WN(0, \sigma^2)$ and $\theta_1, \theta_2, \dots, \theta_q$ are constants.

Previous Questions

TS-13 and 14 (Spectral Analysis)

Q1) Define spectral decomposition, spectral analysis and spectral density of a time series.

Soln:

The spectral decomposition is an analogue for stationary process of the more familiar Fourier representation of deterministic function. It decomposes a time series $\{x_t\}$ into a sum of sinusoidal components with uncorrelated random coefficients.

The analysis of stationary processes by means of their spectral representation is often referred to as the frequency domain analysis of time series or spectral analysis.

Suppose that $\{x_t\}$ is a zero mean stationary time series with autocovariance function $\gamma(\cdot)$ satisfying $\sum_{h=-\infty}^{\infty} |\gamma(h)| < \infty$.

The spectral density of $\{x_t\}$ is the function $f(\cdot)$ defined by

$$f(\lambda) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i\lambda h} \gamma(h), \quad -\pi < \lambda < \pi \quad (1)$$

ARMA(1,1)

$$x_t - \phi x_{t-1} = z_t + \theta z_{t-1} \dots (i)$$

A stationary soln of the ARMA(1,1) equations exists if and only if $\phi \neq \pm 1$

If $|\phi| < 1$, then the unique stationary solution is given by (i). In this case, we say that $\{x_t\}$ is causal on a causal function of $\{z_t\}$, since $\{x_t\}$ can be expressed in terms of the current and past values of z_s , $s \leq t$.

$$x_t = z_t + (\phi + \theta) \sum_{j=1}^{\infty} \phi^{j-1} z_{t-j} \dots (ii)$$

If $|\phi| > 1$, then the unique stationary solution is given by (ii). The soln is noncausal, since x_t is then a function of z_s , $s > t$

$$x_t = -\theta \phi^{-1} z_t - (\phi + \theta) \sum_{j=1}^{\infty} \phi^{-j-1} z_{t+j}$$

where,

$$e^{i\pi} = \cos \pi + i \sin \pi \text{ and } i = \sqrt{-1}$$

The summability of $|x(n)|$ towards $\{x\}$ in equation (1) converges absolutely as,

$$|e^{-ihn}|^2 = \cos^2(hn) + \sin^2(hn) = 1$$

(2) state and prove the basic properties of spectral densities.

Soln:

(a) $f(\cdot)$ is even, i.e., $f(n) = f(-n)$

proof: Since $\sin(\cdot)$ is an odd function and $\cos(\cdot)$ and $\gamma(\cdot)$ are even functions we have,

$$\begin{aligned} f(n) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} (\cos(hn) - i \sin(hn)) \gamma(h) \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(-hn) \gamma(h) + 0 \\ &= f(-n) \end{aligned}$$

(b) $f(n) > 0$ for all $n \in (-\pi, \pi]$

proof:

For each positive integer N define

$$\begin{aligned} f_N(n) &= \frac{1}{2\pi N} E \left(1 \sum_{n=1}^N x_n e^{-in\pi} \sum_{n=1}^N x_n e^{-in\pi} \right) \\ &= \frac{1}{2\pi N} E \left(\sum_{n=1}^N x_n e^{-in\pi} \sum_{n=1}^N x_n e^{-in\pi} \right) \\ &= \frac{1}{2\pi N} \sum_{|h| \leq N} (N - |h|) e^{-ih\pi} \gamma(h) \end{aligned}$$

clearly the function $f_N(\cdot)$ is non-negative for each N , since $f_N(n) \rightarrow 0$ as $N \rightarrow \infty$, $\rightarrow \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\pi} \gamma(h) = f(n)$ as $N \rightarrow \infty$, $f(\cdot)$ must also be non-negative.

$$(c) \gamma(k) = \int_{-\pi}^{\pi} e^{ik\pi} f(n) d\pi = \int_{-\pi}^{\pi} \cos(k\pi) f(n) d\pi$$

proof:

$$\begin{aligned} \int_{-\pi}^{\pi} e^{ik\pi} f(n) d\pi &= \int_{-\pi}^{\pi} \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{i(k-h)\pi} \gamma(h) d\pi \\ &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \gamma(h) \int_{-\pi}^{\pi} e^{i(k-h)\pi} d\pi \\ &= \gamma(k) \end{aligned}$$

✓ (3) How could you use the spectral density to estimate the autocovariance function (ACVF) of a time series process?

Soln:

We know that,

A function $f(\omega)$ is the spectral density of a stationary time series $\{x_t\}$ with ACVF $\gamma(h)$ if

(i) $f(\omega) \geq 0$ for all $\omega \in (0, \pi]$

(ii) $\gamma(h) = \int_{-\pi}^{\pi} e^{ih\omega} f(\omega) d\omega$ for all integers h .

• Spectral representation of ACVF

$\gamma(h)$ is a ACVF of $\{x_t\}$ if

There exists a non decreasing bounded function F on $[-\pi, \pi]$ with

$F(-\pi) = 0$

That is, $\gamma(h) = \int_{(-\pi, \pi]} e^{ih\omega} dF(\omega)$ for all integers h

F is called the spectral distribution function of $\gamma(\cdot)$.

$f(\omega) = \int_{-\pi}^{\pi} \gamma(h) e^{-ih\omega} dh$ for all $\omega \in [-\pi, \pi]$

✓ (4) Check whether the following function is the ACVF of a stationary process,

$$\gamma(h) = \begin{cases} 1 & \text{if } h = 0 \\ -0.5 & \text{if } h = \pm 2 \\ -0.25 & \text{if } h = \pm 3 \\ 0 & \text{if } h = \pm 3 \end{cases}$$

Solution:

We have that $\gamma(h)$ is an even function since $\gamma(h) = \gamma(-h)$ and

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-ih\omega} \gamma(h) \\ &= \frac{1}{2\pi} [-0.25 e^{i3\omega} - 0.5 e^{i2\omega} + 1 - 0.5 e^{-i2\omega} - 0.25 e^{-i3\omega}] \end{aligned}$$

$$= \frac{1}{2\pi} [1 - 0.25 (e^{i3\omega} + e^{-i3\omega}) - 0.5 (e^{i2\omega} + e^{-i2\omega})]$$

$$= \frac{1}{2\pi} [1 - 0.25 \times 2 \cos(3\omega) - 0.5 \times 2 \cos(2\omega)]$$

$$= \frac{1}{2\pi} [1 - 0.5 \cos(3\omega) - \cos(2\omega)]$$

which indicates that $f(\omega)$ is not always non-negative on $\omega \in (-\pi, \pi]$ for instance, $f(\omega) = -\frac{1}{4\pi}$.

Hence, $\{x_t\}$ is not an ARF for a stationary time series.

(5) Find the spectral density of an MA(1) process,

$$x_t = z_t - 0.4z_{t-1}, \text{ where } \{z_t\} \sim WN(0, 1)$$

Hence, find the frequency spectrum of that process and comment.

Soln:

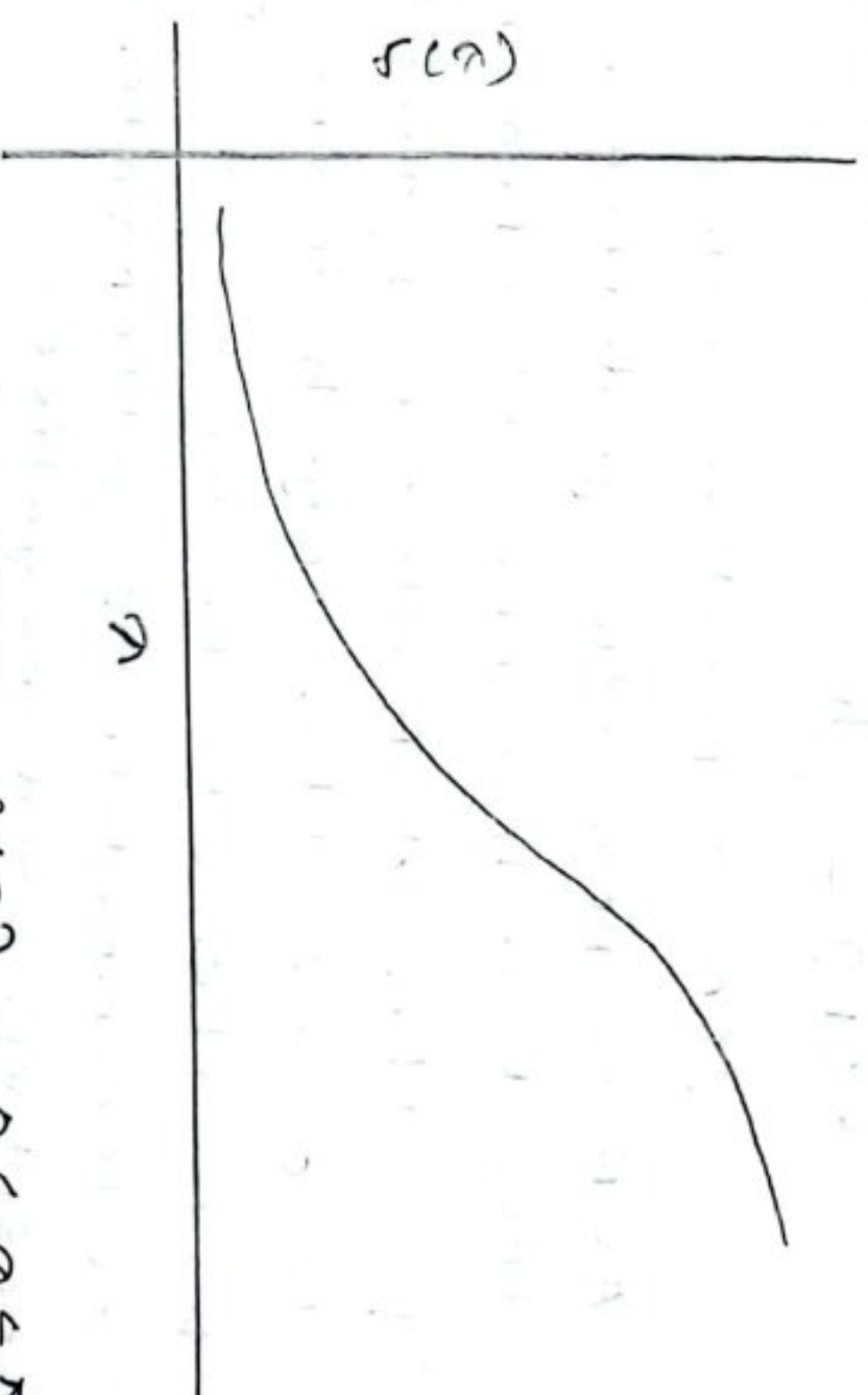
We know that, the spectral density of an MA(1) process is,

$$f(\omega) = \frac{\sigma^2}{2\pi} (1 + 2\theta \cos \omega + \theta^2)$$

$$\therefore f(\omega) = \frac{1}{2\pi} (1 + 2 \times (-0.4) \cos \omega + (-0.4)^2)$$

$$= \frac{1}{2\pi} (1 - 0.8 \cos \omega + 0.16)$$

(6)



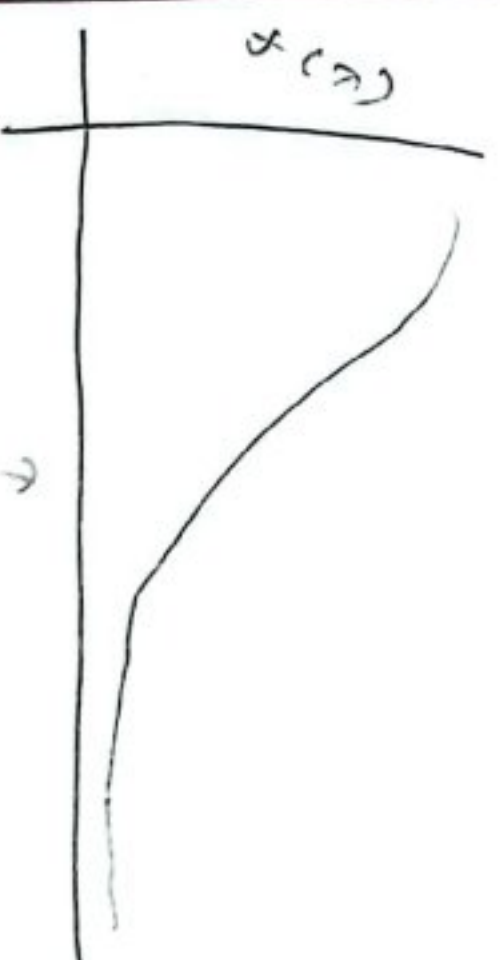
The spectral density $f(\omega)$, $0 \leq \omega \leq \pi$, of $x_t = z_t - 0.4z_{t-1}$, where $\{z_t\} \sim WN(0, 1)$ (slowly increasing function)

(6) Under usual notations, write down the AR(1) process if $\phi = 0.5$.

Soln:

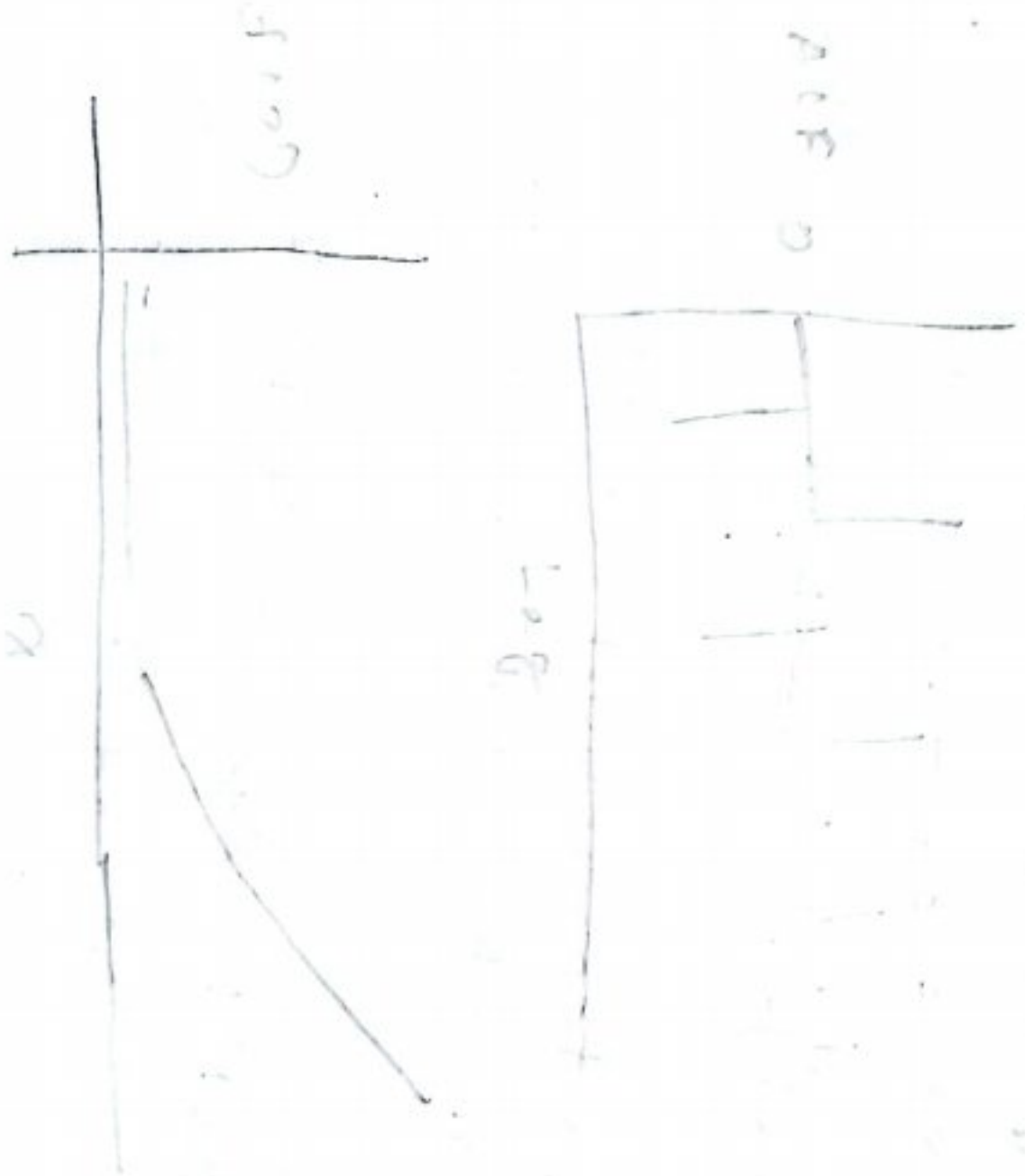
$$\text{AR(1): } x_t = \phi x_{t-1} + z_t$$

$$\therefore x_t = 0.5x_{t-1} + z_t$$



$$\gamma(h) = \rho^h$$

So, for a positive value of ρ , the ACF exponentially decreases to 0 as the lag h increases. For negative ρ , the ACF also exponentially decays to 0 as the lag h increases, but the algebraic signs for the autocorrelations alternate between positive and negative.



✓ (Q) Why Fourier transformation is used in spectral analysis?

Soln:

Because, it allows us to decompose a complex valued function into its frequency components.

$$f(x) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x} dx$$

• $F = \begin{cases} \text{discrete distn} \\ \text{contn distn} \end{cases}$ — Time series have discrete spectrum
Time series

Handwritten notes and diagrams illustrating the relationship between discrete and continuous distributions and their corresponding spectra. It includes a diagram showing a discrete distribution (represented by a series of vertical lines) and a continuous distribution (represented by a smooth curve), with arrows indicating the transformation between them.

The periodogram:

$\{x_t\}$ is a stationary process with ACF $\gamma(k)$ and the spectral density $f(\omega)$.

Periodogram $I_m(\omega)$ is an analogue of $2\pi f(\omega)$

$X = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} \in \mathbb{C}^m \rightarrow$ vector of complex numbers. Set of all column vectors with complex valued components.

Let, $w_k = \frac{2\pi k}{m}$, k is any integer. where $\frac{m-1}{2} \leq k \leq \frac{m}{2}$

f_m , Fourier frequencies associated with sample size m ($f_m \rightarrow$ a subset of $[-\pi, \pi]$)

$e_k = \frac{1}{\sqrt{m}} \begin{bmatrix} e^{i w_k} \\ e^{2i w_k} \\ \vdots \\ e^{i m w_k} \end{bmatrix}$ (m vectors are introduced)

hence, e_i 's are orthogonal as, $e_j^* e_k = \begin{cases} 1, & \text{if } j=k \\ 0, & \text{if } j \neq k \end{cases}$

e_j^* are the row vectors whose k th component is the complex conjugate of the k th component of e_j .

e_i 's are basis for $\mathbb{C}^m$, $x \in \mathbb{C}^m$ and expressed as $x = \sum_k a_k e_k$

$a_k = e_k^* x = \frac{1}{\sqrt{m}} \sum_{t=1}^m x_t e^{-i t w_k}$
 $a_k \rightarrow$ discrete Fourier transform of the series x_t

The periodogram of x_t is the function $I_m(\omega) = \frac{1}{m} \left[\sum_{t=1}^m x_t e^{-i t \omega} \right]^2$

Time-invariant linear Filters

Use:

1) smoothing data

2) Estimating trend and seasonal component.

• Linear filter: $e = \{e_{t,k} : t, k = 0, \pm 1, \dots\}$

$$Y_t = \sum_{k=-\infty}^{\infty} e_{t,k} X_k$$

• The TLF if $e_{t,t-k}$ are independent

$$Y_t = \sum_{k=-\infty}^{\infty} e_{t,t-k} X_{t-k} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-k}$$

$$Y_{t-s} = \sum_{k=-\infty}^{\infty} e_{t-s,t-k} X_{t-s-k} = \sum_{k=-\infty}^{\infty} \psi_k X_{t-s-k}$$

$$\psi_k = e_{t,t-k}$$

• A time shifted linear process

$\{Y_{t-s}, t = 0, \pm 1, \dots\}$ is obtained from $\{X_{t-s}\}$ using linear filter

$$\psi = \{\psi_j, j = 0, \pm 1, \dots\}$$

↓ causal if

$$\psi_j = 0 \text{ for } j < 0$$

Basic properties of $f(\omega)$

(1) $f(\omega)$ is even i.e., $f(\omega) = f(-\omega)$

proof:

$$f(\omega) = \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} e^{-i h \omega} \gamma(h)$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} [\cos(h\omega) - i \sin(h\omega)] \gamma(h)$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(h\omega) \gamma(h) + 0$$

$$= \frac{1}{2\pi} \sum_{h=-\infty}^{\infty} \cos(-h\omega) \gamma(h) \quad \text{since } \gamma(h) = \gamma(-h)$$

$$= f(-\omega)$$

(2) $f(\omega) > 0$ for all $\omega \in (-\pi, \pi]$

proof:

For each positive integer N

$$f_N(\omega) = \frac{1}{2\pi N} \cdot E \left[\left| \sum_{n=1}^N x_n e^{-i n \omega} \right|^2 \right]$$

$$= \frac{1}{2\pi N} \cdot E \left[\sum_{n=1}^N x_n e^{-i n \omega} \sum_{s=1}^N x_s e^{-i s \omega} \right]$$

$$= \frac{1}{2\pi N} \sum_{|h| < N} (N - |h|) e^{-i h \omega} \gamma(h)$$

$$= \frac{1}{2\pi} \sum_{|h| < N} \left(1 - \frac{|h|}{N}\right) e^{-i h \omega} \gamma(h)$$

clearly, the function f_N is non-negative
for each N and since
 $f_N \rightarrow \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} e^{-in\lambda} \gamma(n) = f(\lambda)$ as $N \rightarrow \infty$,
the function f must also be
non-negative.

$$(2) \gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda \\ = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda$$

Proof:

$$\begin{aligned} \gamma(k) &= \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda \\ &= \int_{-\pi}^0 e^{ik\lambda} f(\lambda) d\lambda + \int_0^{\pi} e^{ik\lambda} f(\lambda) d\lambda \\ &= \int_{-\pi}^{\pi} e^{-ik\lambda} f(-\lambda) d\lambda + \int_0^{\pi} e^{ik\lambda} f(\lambda) d\lambda \\ &= \int_0^{\pi} e^{-ik\lambda} f(\lambda) d\lambda + \int_0^{\pi} e^{ik\lambda} f(\lambda) d\lambda \\ &= \int_0^{\pi} (e^{-ik\lambda} + e^{ik\lambda}) f(\lambda) d\lambda \\ &= 2 \int_0^{\pi} \cos(k\lambda) f(\lambda) d\lambda \end{aligned}$$

$$\begin{aligned} &= \int_0^{\pi} \cos(k\lambda) f(\lambda) d\lambda + \int_0^{\pi} \cos(k\lambda) f(\lambda) d\lambda \\ &= \int_{-\pi}^0 \cos(-k\lambda) f(-\lambda) d\lambda + \int_0^{\pi} \cos(k\lambda) f(\lambda) d\lambda \\ &= \int_{-\pi}^0 \cos(k\lambda) f(\lambda) d\lambda + \int_0^{\pi} \cos(k\lambda) f(\lambda) d\lambda \\ &= \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda \end{aligned}$$

Therefore,

$$\gamma(k) = \int_{-\pi}^{\pi} e^{ik\lambda} f(\lambda) d\lambda \\ = \int_{-\pi}^{\pi} \cos(k\lambda) f(\lambda) d\lambda$$

$$g_\alpha(z) = \frac{\sigma_2(1 + \theta_1 z + \dots + \theta_q z^q)}{(1 + \theta_1 z^{-1} + \dots + \theta_q z^{-q})} \cdot \frac{(1 - \varphi_1 z - \dots - \varphi_p z^p)(1 - \varphi_1 z^{-1} - \dots - \varphi_p z^{-p})}{1 - \dots - \varphi_p z^{-p}}$$

TS-12 (Previous Questions)

(c) Obtain the ACF and PACF of this ARMA process.

50305

$$\frac{(CH_3)_2C}{CH_3CH_2} = (CH_3)_2C$$

Calculation of the ACPR:

The ACVP of the seasonal ARMA (CP) is process defined by

$$\phi(B) \times t = \theta(B) \mathbb{Z} + \gamma \quad (1) \quad \{ \mathbb{Z} + \gamma \} \sim N_2(0, \sigma^2)$$

where $\phi(z) = 1 - q(1 - z)^{-1}$ and

Eq(2) = $1 + \sigma_{12} + \dots + \sigma_{n-2,n}$. The causal assumption implies that

$$X_t = \sum_{j=0}^{\infty} \alpha_j Z_{t-j} \quad \dots \quad (1)$$

In the special case, where $\{x, t\}$

is the linear process $x(n) \times b^{-n}$

$$x \cdot x = \sum_{j=-\infty}^{\infty} x_j x_{-j}$$

from (1) and (2) $\Rightarrow \phi = 0$

$$\chi(\text{Ch}) = \sum_{j=0}^n E[\chi + j | \chi \in \gamma] = \sum_{j=0}^{n-1} \frac{\alpha^j}{\alpha - 1} = \frac{\alpha^n - 1}{\alpha - 1}$$

$\gamma_x(z)$

The autocorrelation function:

$$\gamma(h) = \frac{\gamma(h)}{\gamma(0)}$$

For any set of sample observations $\{x_1, \dots, x_m\}$, the sample ACF $\hat{\gamma}(h)$ is computed as

$$\hat{\gamma}(h) = \frac{\hat{\gamma}(h)}{\hat{\gamma}(0)} \quad \text{for } h = 0, 1, \dots, m-1$$

The partial autocorrelation function:

The PACF of an ARMA process $\{x_t\}$ is the function $\alpha(h)$ defined by the correlations

$$\alpha(0) = 1, \quad \alpha(h) = \frac{\gamma(h) - \sum_{j=1}^{h-1} \alpha(j)\gamma(h-j)}{\gamma(0)}$$

and $\alpha(h) = \phi(h)/\phi(0)$

where $\phi(h)$ is the last component

$$\text{of } \Phi_h = [\phi_1, \dots, \phi_h]$$

$$[\phi_1, \dots, \phi_h] = [\gamma(1), \dots, \gamma(h)] \text{ and}$$

$$\gamma_h = [\gamma(1), \dots, \gamma(h)]'$$

For any set of observations $\{x_1, \dots, x_m\}$ with $x_i \neq x_j$ for some i and j , the sample PACF $\hat{\alpha}(h)$ is given by

$$\hat{\alpha}(0) = 1$$

$$\text{and } \hat{\alpha}(h) = \hat{\phi}_h / \hat{\phi}_0, \quad h = 1, 2, \dots, m-1$$

where $\hat{\phi}_h$ is the last component of

$$\hat{\Phi}_h = [\hat{\phi}_1, \dots, \hat{\phi}_h]$$

(2) Discuss the process of estimating the parameters of an ARMA (p, q) process by the maximum likelihood method.

Soln:

The Gaussian likelihood for an ARMA process:

$$L(\phi, \theta, \sigma^2) = \frac{1}{\sqrt{(2\pi\sigma^2)^m}} \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^m \frac{(x_j - \hat{x}_j)^2}{\sigma_j^2} \right\}$$

Maximum likelihood estimators:

$$\hat{\sigma}^2 = m^{-1} S(\hat{\phi}, \hat{\sigma})$$

where,

$$S(\hat{\phi}, \hat{\sigma}) = \sum_{j=1}^m (x_j - \hat{x}_j)^2 / n_j - 1$$

and $\hat{\phi}, \hat{\sigma}$ are the values of ϕ, σ that minimize

$$L(\phi, \sigma) = \sum_{j=1}^m (m-1) S(\phi, \sigma) + m^{-1} \sum_{j=1}^m \ln n_j$$

AIC criterion:

choose p, q, ϕ_p and σ_q to minimize

$$AIC = -2 \ln L(\phi_p, \sigma_q, S(\phi_p, \sigma_q)(m))$$

$$+ 2(p+q+1)m / (m-p-q-2)$$

$$= (S(\phi_p, \sigma_q))^{1/2}$$

$$\left\{ \frac{1}{n} \sum_{j=1}^m \frac{1}{n_j} - \frac{1}{n} \right\}^{1/2}$$

(3) What is meant by diagnostic checking of a time series model?

What are the possible ways for doing

diagnostic checking of a time series model?

Soln:

The goodness of fit of a statistical model to a set of data is judged by comparing the observed values with the corresponding predicted values obtained from the fitted model.

If the fitted model is appropriate, then the residuals should behave in a manner that is consistent with the model.

The residuals are then,
$$w_t = [x_t - \hat{x}_t(\hat{\phi}, \hat{\sigma})] / [\pi_t(\hat{\phi}, \hat{\sigma})]^{1/2}$$

where $\{x_t\}$ is the sequence of observations $t=1, \dots, m$

In particular, the sequence $\{w_t\}$ should be approximately

- (1) uncorrelated $\{w_t\} \sim WN(0, \sigma^2)$
- (2) independent $\{w_t\} \sim IID(0, \sigma^2)$
- (3) normally distributed $\{w_t\} \sim N(0, \sigma^2)$

gac

The rescaled residuals

$$\hat{r}_t = \hat{w}_t / \hat{\sigma}$$

Possible ways:

- The graph of the rescaled residuals
- The sample ACF of the residuals
- Tests for randomness of the residuals

Note: Recall testing estimated noise sequence.

* Order selection (AIC)

The FPE criterion:

The FPE criterion was developed by Akaike (1969) to select the appropriate order of an AR process to fit to a time series $\{x_t\}$.

$$FPE_p = \hat{\sigma}^2 \frac{n+p}{n-p}$$

Choose the value of p that minimizes FPE.

$$AIC(p) = -2 \ln L(x(\beta), Sx(\beta)/m) + 2(p+1)$$

$$AIC(p) = -2 \ln L(x(\beta), Sx(\beta)/m) + 2(p+1)m/(m-p-1)$$

Q4) What is a seasonal ARIMA (SARIMA) model?

Soln:

- ARIMA is one of the most widely used forecasting methods for univariate time series data with a trend, it does not support time
- The method can handle data with a trend, it does not support time series with a seasonal component.
- An extension to ARIMA that supports the direct modeling of the seasonal component of the series is called SARIMA.

Mathematical representation of ARIMA to SARIMA:

- If d is a non-negative integer, then $\{x_t\}$ is an ARIMA(p, d, q) process if $y_t = (1-B)^d x_t$ is a causal ARMA(p, q) process.

That means,

$$\Phi(B)Y_t = \Phi(B)(1-B)^d x_t = \Theta(B)Z_t$$

- If we fit an ARMA(p, q) model $\Phi(B)Y_t = \Theta(B)Z_t$ to the differenced series $y_t = (1-B)^d x_t$, then the model for the original series is $\Phi(B)(1-B)^d x_t = \Theta(B)Z_t$.

This is a special case of the general SARIMA model defined as follows:

$$\Phi(B)\Phi(B^s)(1-B)^d(1-B^s)^d x_t = \Theta(B)\Theta(B^s)Z_t$$

- If d and D are non-negative integers, then $\{x_t\}$ is a seasonal ARIMA(p, d, q) if the differenced $y_t = (1-B)^d x_t$ is an ARIMA(p, d, q) process defined by $\Phi(B)\Phi(B^s)(1-B)^d(1-B^s)^d x_t = \Theta(B)\Theta(B^s)Z_t$ with $\Phi(B^s)$ and $\Theta(B^s)$ defined by $\Phi(B^s) = 1 - \phi_1 B^s - \dots - \phi_P B^{Ps}$ and $\Theta(B^s) = 1 + \theta_1 B^s + \dots + \theta_Q B^{Qs}$.

where $\Phi(B^s) = 1 - \phi_1 B^s - \dots - \phi_P B^{Ps}$

$$\Theta(B^s) = 1 + \theta_1 B^s + \dots + \theta_Q B^{Qs}$$

$$\Phi(B)\Phi(B^s)(1-B)^d(1-B^s)^d x_t = \Theta(B)\Theta(B^s)Z_t$$

The rescaled residuals

$$\hat{r}_t = \hat{w}_t / \hat{\sigma}$$

Possible ways:

- The graph of the rescaled residuals
- The sample ACF of the residuals
- Tests for normality of the residuals

Note: Recall testing estimated noise sequence.

* Order selection (ARCP)

The FPE criterion:

The FPE criterion was developed by Akaike (1969) to select the appropriate order of an AR process to fit to a time series $\{x_t\}$.

$$FPE_p = \hat{\sigma}^2 \frac{n+p}{n-p}$$

Choose the value of p that minimizes FPE_p .

$$AIC(p) = -2 \ln L(x; \beta, Sx(p)/m) + 2(p+1)$$

$$AIC(p) = -2 \ln L(x; \beta, Sx(p)/m) + 2(p+1)m/(m-p-2)$$

Q4 What is a seasonal ARIMA (SARIMA) model?

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That means, $\Phi(B) \Psi(B) x_t = \Theta(B) z_t$
 $\Phi(B) y_t = \Phi(B) (1-B)^d x_t = \Theta(B) z_t$

- If we fit an ARMA(p, q) model $\Phi(B) y_t = \Theta(B) z_t$ to the differenced series $y_t = (1-B)^d x_t$, then the model for the original series is $\Phi(B) (1-B)^d x_t = \Theta(B) z_t$.

This is a special case of the general SARIMA model defined as follows:

- If d and D are non-negative

integers, then $\{x_t\}$ is a seasonal ARIMA(p, d, q) $\times$ (P, D, Q)s process with period s if the differenced series $y_t = (1-B)^d (1-B^s)^D x_t$ is a causal ARMA process defined by

$$\Phi(B) \Phi(B^s) y_t = \Theta(B) \Theta(B^s) z_t$$

where $\Phi(B) = 1 - \phi_1 B - \dots - \phi_p B^p$,

$$\Phi(B^s) = 1 - \phi_1 B^s - \dots - \phi_p B^{ps}$$

$$\Theta(B) = 1 + \theta_1 B + \dots + \theta_q B^q$$

$$\Theta(B^s) = 1 + \theta_1 B^s + \dots + \theta_q B^{qs}$$

(5) When do you prefer ARIMA to ARMA model?

Soln:

We should prefer ARIMA models over ARMA models when dealing with non-stationary time series data + exhibit trends, seasonality, or require differencing to achieve stationarity.

PAEF of an ARCP

$$\alpha(p) = \phi_p$$

$$\text{and } \alpha(h) = 0 \text{ for } h > p$$

PAEF of an MA(1)

$$\alpha(h) = \theta_h = \frac{-(-\theta)^h}{1 + \theta + \dots + \theta^2h}$$

(6) In preliminary estimation of an ARMA process which method is useful for pure autoregressive model and why?

Soln:

For a pure autoregressive model, the PAEF is useful. Because, it helps identify the order of the AR process by capturing direct influences from past values while eliminating shorter lag influences.

Note:

If there are significant spikes at multiple lags, it suggests an MA process. (ACE)

$$\nabla^2(X_t) = \nabla[\nabla(X_t)]$$

$$= \nabla[(1-B)X_t]$$

$$= (1-B)(1-B)X_t$$

$$= (1-2B+B^2)X_t$$

$$= X_t - 2BX_t + B^2X_t$$

$$= X_t - 2X_{t-1} + X_{t-2}$$

$$\nabla^2(X_t) = (1-B)^2 X_t$$

$$\nabla^i(X_t) = (1-B)^i X_t$$

⑦ Difference operator (Δ)

$$1 - B = 1 - B$$

$$1 - BX_t = (1-B)X_t$$

$$(1-B)X_t = X_t - BX_t$$

$$(1-B)X_t = X_t - BX_t$$

$$\Delta(X_t) = (1-B)X_t$$

$$X_t - BX_t$$

• strictly stationary time series

A time series $\{X_t, t = 0, \pm 1, \dots\}$ is said to be strictly stationary if it has stationary properties similar to those of the "time-shifted" series $\{X_{t+h}, t = 0, \pm 1, \dots\}$ for each integer h .

⊗ Backward shift operator (B)

$$X_t = \{X_1, X_2, X_3, X_4, \dots\}; \quad t \geq 1$$

$$B(X_t) = X_{t-1}$$

$$\begin{aligned} B^2(X_t) &= B[B(X_t)] \\ &= B[X_{t-1}] \\ &= X_{t-2} \end{aligned}$$

$$B^q(X_t) = X_{t-q}$$

⊗ Difference operator (∇)

$$\nabla X_t = X_t - X_{t-1}$$

$$B(X_t) = X_{t-1}$$

$$\begin{aligned} \therefore \nabla X_t &= X_t - B(X_t) \\ &= X_t(1 - B) \end{aligned}$$

$$\nabla^j(X_t) = \nabla[\nabla^{j-1}(X_t)] \quad \left\{ \begin{array}{l} j \geq 1 \\ \nabla^0(X_t) = X_t \end{array} \right.$$

$$\nabla^0(X_t) = X_t$$

$$h > 3, \quad r(h, 3) = 0$$

Similarly,

$$r(-1) = \frac{2\sigma\omega^2}{9}$$

$$r(-2) = \frac{\sigma\omega^2}{9}$$

$$r(-3) = 0$$

$$\therefore r(s, t) = \begin{cases} \frac{1}{3}\sigma\omega^2 & ; h=0 \\ \frac{2}{9}\sigma\omega^2 & ; h=\pm 1 \quad [|h|=1] \\ \frac{1}{9}\sigma\omega^2 & ; h=\pm 2 \quad [|h|=2] \\ 0 & ; h > \pm 3 \quad [|h| > 3] \end{cases}$$

This autocovariance function is independent of t .

$\therefore \{v_t\}$ is a stationary process.

For stationary process,

$$g(s, t) = g(h) = \frac{r(h)}{r(0)}$$

$$\therefore g(h) = \begin{cases} 1 & ; h=0 \\ 2/3 & ; |h|=1 \\ 1/3 & ; |h|=2 \\ 0 & ; |h| > 3 \end{cases}$$

$$= \frac{1}{9} \times 3 \sigma \omega^2$$

$$= \frac{\sigma \omega^2}{3}$$

$$\therefore \gamma(h) = \gamma(0) = \frac{\sigma \omega^2}{3}$$

$$\text{when } h=1, \quad h = s - t$$

$$\Rightarrow s = t + h$$

$$\Rightarrow s = t + 1$$

$$\begin{aligned} \gamma(1) &= \frac{1}{9} \{ E[\omega_{t-1} \omega_t] + E[\omega_{t-1}, \omega_{t+1}] \\ &\quad + E[\omega_{t-1} \omega_{t+2}] + E[\omega_t \cdot \omega_t] \\ &\quad + E[\omega_t \cdot \omega_{t+1}] + E[\omega_t \cdot \omega_{t+2}] \\ &\quad + E[\omega_{t+1} \cdot \omega_t] + E[\omega_{t+1}, \omega_{t+1}] \\ &\quad + E[\omega_{t+1} \cdot \omega_{t+2}] \} \end{aligned}$$

$$= \frac{1}{9} \{ E[\omega_t \cdot \omega_t] + E[\omega_{t+1}, \omega_{t+1}] \}$$

$$= \frac{1}{9} (2 \sigma \omega^2)$$

$$= \frac{2}{9} \sigma \omega^2$$

$$\begin{aligned} \text{Similarly, } \gamma(2) &= \frac{1}{9} [E(\omega_{t+1}, \omega_{t+1})] \\ &= \frac{1}{9} \sigma \omega^2 \end{aligned}$$

$$\gamma(3) = 0$$

when $s = t$

$$h = s - s = t - t = 0$$

$$v_t = \frac{1}{3} [\omega_{t-1} + \omega_t + \omega_{t+1}]$$

$$v_s = \frac{1}{3} [\omega_{s-1} + \omega_s + \omega_{s+1}]$$

$$\gamma(h) = E \left[\left\{ \frac{1}{3} (\omega_{t-1} + \omega_t + \omega_{t+1}) \right\} \times \left\{ \frac{1}{3} (\omega_{s-1} + \omega_s + \omega_{s+1}) \right\} \right]$$

$$= \frac{1}{9} \left\{ E[\omega_{t-1} \omega_{s-1}] + E[\omega_{t-1} \omega_s] + E[\omega_{t-1} \omega_{s+1}] + E[\omega_t \omega_{s-1}] + E[\omega_t \omega_s] + E[\omega_t \omega_{s+1}] + E[\omega_{t+1} \omega_{s-1}] + E[\omega_{t+1} \omega_s] + E[\omega_{t+1} \omega_{s+1}] \right\}$$

when $s = t$

$$\gamma(h) = \frac{1}{9} \left\{ E[\omega_{t-1} \omega_{t-1}] + E[\omega_{t-1} \omega_t] + E[\omega_{t-1} \omega_{t+1}] + E[\omega_t \omega_{t-1}] + E[\omega_t \omega_t] + E[\omega_t \omega_{t+1}] + E[\omega_{t+1} \omega_{t-1}] + E[\omega_{t+1} \omega_t] + E[\omega_{t+1} \omega_{t+1}] \right\}$$

$$= \frac{1}{9} \left\{ E[\omega_{t-1} \omega_{t-1}] + E[\omega_{t+1} \omega_{t+1}] + E[\omega_t \omega_t] \right\}$$

$$= \frac{1}{9} \left\{ \sigma_\omega^2 + \sigma_\omega^2 + \sigma_\omega^2 \right\}$$

(3) Find the autocovariance function of moving average process is given by

$$v_t = \frac{1}{3} (\omega_{t-1} + \omega_t + \omega_{t+1}) : \omega_t \sim (0, \sigma_\omega^2)$$
 where ω_t is an uncorrelated process.
 check stationarity of the series.

$\Rightarrow$ first, we have to show μ_t is independent of time t .

$$\begin{aligned} \mu_t &= E[v_t] = E\left[\frac{1}{3} (\omega_{t-1} + \omega_t + \omega_{t+1})\right] \\ &= \frac{1}{3} \{E[\omega_{t-1}] + E[\omega_t] + E[\omega_{t+1}]\} \end{aligned}$$

Since, $E[\omega_t] = 0$ then

$$E[\omega_{t+1}] = 0 \text{ and } E[\omega_{t-1}] = 0$$

$$\therefore \mu_t = \frac{1}{3} [0 + 0 + 0] = 0$$

$\therefore \mu_t$ is a constant and independent of time t .

Now, we have to find the autocovariance function,

$$\begin{aligned} \gamma(s, t) &= \gamma(h) \\ &= E[(v_s - \mu_s)(v_t - \mu_t)] \\ &= E[v_t * v_s] \end{aligned}$$

$$s - t = h$$

$$\text{when } s = t ; h = s - s = 0$$

$$\gamma(s, t) = \gamma(h) = \gamma(0) = E[\omega_s, \omega_s] \\ = E[\omega_t, \omega_t]$$

$$\Rightarrow \gamma(s, t) = \text{variance} \\ = \sigma_\omega^2$$

$$\text{when } s \neq t ;$$

$$\gamma(s, t) = E[\omega_s, \omega_t]$$

$$= 0 \quad [\text{since, they are uncorrelated}]$$

- Two real valued random variables are said to be uncorrelated, if their covariance is zero.

$\therefore$ The autocovariance function, can be written as,

$$\gamma(s, t) = \begin{cases} \sigma_\omega^2 ; & \text{if } h = 0 \\ 0 ; & \text{o/w} \end{cases}$$

$\therefore$ Autocovariance function is independent of time t .

$\therefore \omega_t$ is stationary process.

(1) Find the auto covariance function of w_t which is uncorrelated and has zero mean. Show that the process is stationary.

$$w_t \sim (0, \sigma_w^2)$$

Soln:

Here, $\mu = 0$, $\sigma^2 = \sigma_w^2$

and w_t is uncorrelated.

First, we have to show μ_t is independent of time t .

$$\mu_t = E(w_t)$$

$$\Rightarrow \mu_t = 0$$

$\therefore \mu_t$ is a constant and independent of time t .

Now, we have to find autocovariance function.

$$\gamma(s, t) = \gamma(h)$$

$$= \text{cov}(w_s, w_t)$$

$$= E[(w_s - \mu_s)(w_t - \mu_t)]$$

$$\text{But, } \mu_t = \mu_s = 0$$

$$\therefore \gamma(s, t) = E[(w_s - 0)(w_t - 0)]$$
$$= E[w_s, w_t]$$

Tutorial 3

Jahangirnagar University
Department of Statistics
Savar, Dhaka-1342

Date: 02.10.2023

Course code: STAT-306

Course Title: Time Series Analysis

Full marks: 20

Answer the following questions.

- Q1. What is a linear process? Give some examples of such process. When an ARMA process is considered as, i) a causal process and ii) an invertible process?
- Q2. When would you prefer an ARIMA process to an ARMA process? Explain with a suitable example. Distinguish between an ARIMA and a SARIMA process.
- Q3. What is meant by the diagnostic checking of a fitted model? How would you proceed for such checking? Explain.
- Q4. Write the basic properties of a spectral density. "All the ACVF functions do not have the spectral density"—Justify. Express the spectral density of an ARMA(p, q) process.

~~100-100?~~
~~100-100?~~
~~100-100?~~

